Lecture 12 Caps & RC Circuits

10-23-11
Midterm Grades

- Mean = 80, Median = 83, Mode = 95
- Standard Deviation = 14
Description of a Capacitor

• A capacitor is a passive element designed to store energy in its electric field.

• A capacitor consists of two conducting plates separated by an insulator (or dielectric).

\[ U_E = \frac{1}{2} \int_V \vec{D} \cdot \vec{E} \, dV \]
Definition of Capacitance

- **Capacitance** \( C \) is the ratio of the charge \( q \) on one plate of a capacitor to the voltage difference \( v \) between the two plates, measured in farads (F).

\[
q = C \ v \quad \text{and} \quad C = \frac{\varepsilon A}{d}
\]

- Where \( \varepsilon \) is the permittivity of the dielectric material between the plates, \( A \) is the surface area of each plate, \( d \) is the distance between the plates.
- Unit: F, pF (\( 10^{-12} \)), nF (\( 10^{-9} \)), and \( \mu \)F (\( 10^{-6} \))
Some Typical Capacitors
Voltage across a Capacitor

• If $i$ is flowing into the +v terminal of C
  – Charging $\Rightarrow i$ is positive ($> 0$)
  – Discharging $\Rightarrow i$ is $- (< 0$)

• The current-voltage relationship of capacitor according to above convention is

$$i = C \frac{d v}{d t}$$

and

$$v = \frac{1}{C} \int_{t_0}^{t} i \; d t + v(t_0)$$
Energy Stored in a Capacitor

• The energy, $w$, stored in the capacitor is

$$w = \frac{1}{2} C v^2$$

• A capacitor is
  – an **open circuit** to dc ($dv/dt = 0$).
  – its voltage **cannot change abruptly** (depends on integral of $i$).
Voltage across a Capacitor

Example

The voltage across a 5-\(\mu\)F capacitor is

\[ v(t) = 10 \cos(6000 \, t) \, \text{V}. \]

Calculate the current through it

\[ i = C \frac{d}{dt} v \]

\[ v = \frac{1}{C} \int_{t_0}^{t} i \, dt + v(t_0) \]
Parallel Capacitors

- The equivalent capacitance of $N$ parallel-connected capacitors is the sum of the individual capacitances.

\[ C_{eq} = C_1 + C_2 + \ldots + C_N \]
Series Capacitors

• The equivalent capacitance of \( N \) series-connected capacitors is the reciprocal of the sum of the reciprocals of the individual capacitances.

\[
\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_N}
\]
Series and Parallel Capacitors

Example 3
Find the equivalent capacitance seen at the terminals of the circuit in the circuit shown below:

Answer:
$C_{eq} = 40 \mu F$
Source-Free RC Circuit

- A **first-order circuit** is characterized by a first-order differential equation.

\[ i_R + i_C = 0 \]

\[ \frac{v}{R} + C \frac{dv}{dt} = 0 \]

- Apply Kirchhoff’s laws to **purely resistive circuit** results in **algebraic equations**.
- Apply the laws to **RC (and RL) circuits** produces **differential equations**.
Source-Free Response of an RC Circuit

- The **natural response** of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with **no external sources of excitation**.

- The **time constant** $\tau$ of a circuit is the time required for the response to decay by a factor of $1/e$ or 36.8% of its initial value.
- $v$ decays **faster for small $\tau$ and slower for large $\tau$**.
First Order DE Solution for RC Circuit

- Use \( \frac{v}{R} + C \frac{dv}{dt} = 0 \)
- Manipulate equation
- Integrate
- Manipulate
- Exponentiate
- Define time constant

\[
\ln v = -\frac{t}{RC} + \ln A
\]

and

\[
\ln \left( \frac{v}{A} \right) = -\frac{t}{RC}
\]

and exponentiation both sides

\[
v(t) = V_o \ e^{-\frac{t}{RC}} = V_o \ e^{-\frac{t}{\tau}}
\]

where

\[
\tau = RC
\]
Solution of DE for Source-Free RC Circuit

The key to working with a source-free RC circuit is finding:

\[ v(t) = V_0 e^{-t/\tau} \]

where \( \tau = R C \)

1. The initial voltage \( v(0) = V_0 \) across the capacitor.
2. The time constant \( \tau = RC \).
Example 1

Refer to the circuit below, determine \( v_C \), \( v_x \), and \( i_o \) for \( t \geq 0 \).
Assume that \( v_C(0) = 30 \) V.

Answer: \( v_C = 30e^{-0.25t} \) V; \( v_x = 10e^{-0.25t} \); \( i_o = -2.5e^{-0.25t} \) A
Source-Free RC Circuit Example

Example 2
The switch in circuit below is opened at $t = 0$, find $v(t)$ for $t \geq 0$.

![Circuit Diagram]

- Please refer to lecture or textbook for more detail elaboration.
Answer: $V(t) = 8e^{-2t} \text{ V}$
Unit-Step Function

• The **unit step function** $u(t)$ is 0 for negative values of $t$ and 1 for positive values of $t$.

\[
\begin{align*}
  u(t) &= \begin{cases} 
  0, & t < 0 \\
  1, & t > 0 
  \end{cases} 
\end{align*}
\]

\[
\begin{align*}
  u(t - t_o) &= \begin{cases} 
  0, & t < t_o \\
  1, & t > t_o 
  \end{cases} 
\end{align*}
\]

\[
\begin{align*}
  u(t + t_o) &= \begin{cases} 
  0, & t < -t_o \\
  1, & t > -t_o 
  \end{cases} 
\end{align*}
\]
Abrupt Change (Switch on or off)

Represent an abrupt change for:

1. voltage source.

2. for current source:
DE for Step-Response of a RC Circuit

- The **step response** of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.

- **Initial condition:**
  \[ v(0-) = v(0+) = V_0 \]

- Applying KCL,
  \[
  c \frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0
  \]
  or
  \[
  \frac{dv}{dt} = - \frac{v - V_s}{RC} u(t)
  \]

- Where \( u(t) \) is the **unit-step function**
Step-Response of a RC Circuit

- Integrating both sides and considering the initial conditions, the solution of the equation is:

\[ v(t) = \begin{cases} 
V_0 & t < 0 \\
V_s + (V_0 - V_s)e^{-t/\tau} & t > 0 
\end{cases} \]

Final value at \( t \to \infty \)  
Initial value at \( t = 0 \)  
Source-free Response

Complete Response = Natural response (stored energy) + Forced Response (independent source)

\[ = V_0e^{-t/\tau} + V_s(1-e^{-t/\tau}) \]
Finding the Step-Response of a RC Circuit

Three steps to find out the step response of an RC circuit:

1. The initial capacitor voltage \( v(0) \).
2. The final capacitor voltage \( v(\infty) \) — DC voltage across C.
3. The time constant \( \tau \).

\[
 v(t) = v(\infty) + [v(0+) - v(\infty)] e^{-t/\tau}
\]

Note: The above method is a short-cut method. You may also determine the solution by setting up the circuit formula directly using KCL, KVL, ohms law, capacitor and inductor VI laws.