Lecture 21
Frequency Response:
Resonance, 2nd Order Filters and Active Filters

Nov. 21, 2011

Frequency Response
Chapter 14 in A & S

• Resonance
• Second Order Filters
• Active Filters
Series Resonance

The quality factor is the ratio of its resonant frequency to its bandwidth.

If the bandwidth is narrow, the quality factor of the resonant circuit must be high.

If the band of frequencies is wide, the quality factor must be low.

The relationship between the B, Q and $\omega_0$:

$$Q = \frac{\text{Peak energy stored in the circuit}}{\text{Energy dissipated by the circuit in one period at resonance}} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

$$B = \frac{R}{L} = \frac{\omega_0}{Q} = \omega_0^2 CR$$
Series Resonance

http://www.intuitior.com/resonance/circuits.html
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Parallel Resonance

It occurs when imaginary part of \( Y \) is zero

\[ Y = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right) \]

Resonance frequency:

\[ \omega_o = \frac{1}{\sqrt{LC}} \text{ rad/s} \quad \text{or} \quad f_o = \frac{1}{2\pi \sqrt{LC}} \text{ Hz} \]
Parallel Resonance

Quality factor,

\[ Q = \frac{\text{Peak energy stored in the circuit}}{\text{Energy dissipated by the circuit in one period at resonance}} = \omega_o RC = \frac{R}{\omega_o L} \]

The relationship between the B, Q and \( \omega_o \):

\[ B = \frac{1}{RC} = \frac{\omega_o}{Q} \]

- The quality factor is the \textit{ratio} of its \textit{resonant frequency} to its \textit{bandwidth}.
- If the bandwidth is \textit{narrow}, the quality factor of the resonant circuit must be \textit{high}.
- If the band of frequencies is \textit{wide}, the quality factor must be \textit{low}.
Series Resonance

Quality factor, $Q = \frac{\text{Peak energy stored in the circuit}}{\text{Energy dissipated by the circuit in one period at resonance}} = \frac{\omega_o L}{R} = \frac{1}{\omega_o CR}$

The relationship between the $B, Q$ and $\omega_o$:

- The quality factor is the ratio of its resonant frequency to its bandwidth.
- If the bandwidth is narrow, the quality factor of the resonant circuit must be high.
- If the band of frequencies is wide, the quality factor must be low.

$B = \frac{R}{L} = \frac{\omega_o}{Q} = \omega_o^2 CR$
Resonance is a condition in an RLC circuit in which the capacitive and inductive reactance are equal in magnitude, thereby resulting in purely resistive impedance.

<table>
<thead>
<tr>
<th>characteristic</th>
<th>Series circuit</th>
<th>Parallel circuit</th>
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<tbody>
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<td>$\omega_o$</td>
<td>$\frac{1}{\sqrt{LC}}$</td>
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<tr>
<td>$Q$</td>
<td>$\frac{\omega_o L}{R}$ or $\frac{1}{\omega_o R C}$</td>
<td>$\frac{R}{\omega_o L}$ or $\omega_o R C$</td>
</tr>
<tr>
<td>$B$</td>
<td>$\frac{\omega_o}{Q}$</td>
<td>$\frac{\omega_o}{Q}$</td>
</tr>
<tr>
<td>$\omega_1$, $\omega_2$</td>
<td>$\omega_o \sqrt{1 + \left(\frac{1}{2Q}\right)^2} \pm \frac{\omega_o}{2Q}$</td>
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<td>$Q \geq 10$, $\omega_1$, $\omega_2$</td>
<td>$\omega_o \pm \frac{B}{2}$</td>
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Ideal Filters
Noise Rejection
FDMA Wireless Systems

Transmitter 1 modulates source output into signal with 200 kHz bandwidth and a center frequency of 92.1 MHz.

Transmitter 2 modulates source output into signal with 200 kHz bandwidth and a center frequency of 92.3 MHz.

Transmitter 3 modulates source output into signal with 200 kHz bandwidth and a center frequency of 92.5 MHz.

Channel capable of passing 92.0–92.6 MHz frequency band

Bandpass filter
- $f_l = 92.0 \text{ MHz}$
- $f_h = 92.2 \text{ MHz}$

Receiver

Demodulator

User A

Bandpass filter
- $f_l = 92.2 \text{ MHz}$
- $f_h = 92.4 \text{ MHz}$

Receiver

Demodulator

User B

Bandpass filter
- $f_l = 92.4 \text{ MHz}$
- $f_h = 92.6 \text{ MHz}$

Receiver

Demodulator

User C

Common shared bandwidth

Signal from Source A to User A

92.0

Signal from Source B to User B

92.2

Signal from Source C to User C

92.4

92.6

frequency in MHz

http://zone.ni.com/devzone/cda/ph/p/id/269
### Relative Advantages of Passive and Active Filters

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<th>Filter Type</th>
<th>Advantages</th>
<th>Limitations</th>
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<td>Stable and Reliable</td>
<td>Requires inductors (size &amp; weight) for pass and stop filters</td>
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<td>Less vulnerable to environment</td>
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<td>Small and light and cheap, Easily implemented in ICs</td>
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• First order filters (R&C or R&L)
  • Low pass
  ![Low pass filter diagram](image)
  • High pass
  ![High pass filter diagram](image)

• Second order filters (all 3: R, L and C)
  • Band pass
  ![Band pass filter diagram](image)
  • Band stop (or reject)
  ![Band stop filter diagram](image)

There are also 2nd order high & low pass filters
Second Order Low-Pass Filter

\[ V_{out} = \frac{Z_C}{Z_R + Z_L + Z_C} \]

\[ V_{in} = \frac{-j}{2\pi fC} \]

\[ V_{in} = \frac{j}{R + j2\pi fL - \frac{j}{2\pi fC}} \]

\[ V_{in} = \frac{-j}{2\pi fRC} \]

\[ V_{out} = H(f) = \frac{-j}{2\pi fRC} \]

\[ \frac{V_{out}}{V_{in}} = \frac{1 + j\frac{2\pi f_0 L}{R} \left( \frac{f}{f_0} - \frac{1}{2\pi f_0 LC} \right)}{1 + jQ_s \left( \frac{f}{f_0} - \frac{f_0}{f} \right)} \]

where \( Q_s = \frac{1}{\omega_0 CR} = \frac{1}{2\pi f_0 CR} \)

and \( 2\pi f_0 = \omega_0 \)
Second-Order Low-Pass Filter

\[ H(f) = \frac{V_{out}}{V_{in}} = \frac{-jQ_s(f_0/f)}{1 + jQ_s(f/f_0 - f_0/f)} \]

\[ = \frac{Q_s(f_0/f) \angle -90^\circ}{\sqrt{1 + Q_s^2(f/f_0 - f_0/f)^2} \angle Tan^{-1} Q_s(f/f_0 - f_0/f)} \]

\[ |H(f)| = \frac{Q_s(f_0/f)}{\sqrt{1 + Q_s^2(f/f_0 - f_0/f)^2}} \]
2nd Order Low-Pass Filter
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Channel capable of passing 92.0-92.6 MHz frequency band.

Common shared bandwidth

Signal from Source A to User A

Signal from Source B to User B

Signal from Source C to User C

http://zone.ni.com/devzone/cda/ph/p/id/269
Second Order High-Pass Filter

At low frequency the capacitor is an open circuit.

At high frequency the capacitor is a short and the inductor is open.

(a) Circuit diagram

(b) Transfer-function magnitude
Second Order Band-Pass Filter

At low frequency the capacitor is an open circuit \((j\omega C)\)

At high frequency the inductor is an open circuit \((j\omega L)\)

(a) Circuit diagram

(b) Transfer-function magnitude
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Common shared bandwidth

Signal from Source A to User A

Signal from Source B to User B

Signal from Source C to User C

frequency in MHz

92.0 92.2 92.4 92.6

http://zone.ni.com/devzone/cda/ph/p/id/269
Second Order Bandstop Filter

At low frequency the capacitor is an open circuit \((j/ωC)\)

At high frequency the inductor is an open circuit \((jωL)\)

(a) Circuit diagram

(b) Transfer-function magnitude

\[
\frac{|H(f)|}{(dB)}
\]
Example

Design a filter that passes frequency components higher than 1 kHz and rejects components lower than 1 kHz. Choose $L=50$ mH

$$f_0 = 1 \text{ kHz} = \frac{1}{2\pi\sqrt{LC}} \quad \Rightarrow \quad C = \frac{1}{(2\pi)^2 f_0^2 L} = \frac{1}{(2\pi)^2 (1 \times 10^3)^2 (50 \times 10^{-3})} = 0.507 \mu F$$
High Pass Filter Example

To avoid amplifying the signal at $f_0$ choose $Q_s=1$
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First-Order Low-Pass Filter

A low-pass filter with a dc gain of \(-\frac{R_f}{R_i}\)
A high-pass filter with a high frequency gain of \(-\frac{R_f}{R_i}\)

First-Order High-Pass Filter

\[
H(f) = \frac{v_o}{v_i} = \frac{Z_f}{Z_i}
\]

\[
Z_i = R_i + \frac{1}{j2\pi f C_i}
\]

\[
Z_f = R_f
\]

\[
H(f) = -\frac{Z_f}{Z_i} = -\frac{R_f}{R_i + \frac{1}{j2\pi f C_i}}
\]

\[
= -\frac{j2\pi f R_f C_i}{1 + j2\pi f R_i C_i} = -\left(\frac{R_f}{R_i}\right)\frac{j2\pi f R_f C_i}{1 + j2\pi f R_i C_i}
\]

\[
= -\left(\frac{R_f}{R_i}\right)\left[\frac{j(f/f_B)}{1 + j(f/f_B)}\right]
\]

\[
f_B = \frac{1}{2\pi R_i C_i}
\]

\[2\pi f_B = \omega_c\]
Higher Order Filters

\[ H(f) = H_1(f)H_2(f)\cdots H_n(f) \]

\[ = (-1)^n \left( \frac{R_f}{R_i} \right)^n \left[ \frac{1}{1 + j(f / f_B)} \right]^n \]
Butterworth Transfer Function

Butterworth filters are characterized by having a particularly flat pass-band.

\[
|H(f)| = \frac{H_0}{\sqrt{1 + (f/f_B)^{2n}}}
\]