Lecture 14
Second Order Transient Response
Damped Harmonic Motion

**Mechanical oscillator**

\[ m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = f_m(t) \]

**Electrical oscillator**

\[ L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = v_s(t) \]
The Series RLC Circuit and its Mechanical Analog

- Drive voltage \( v_S(t) \)
- Charge \( q \)
- Current \( i \)
- Inductance \( L \)
- Inductive Energy \( U = (1/2)Li^2 \)
- Capacitance \( C \)
- Capacitive Energy \( U = (1/2)(1/C)q^2 \)
- Resistance \( R \)
- Applied force \( f(t) \)
- Displacement \( x \)
- Velocity \( v \)
- Mass \( m \)
- Inertial Energy \( (1/2)mv^2 \)
- Spring constant \( 1/k \)
- Spring energy \( U = (1/2)kx^2 \)
- Dampening constant \( b \)
Second-Order Circuits

\[ L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int_0^t i(t)dt + v_c(0) = v_s(t) \]

Differentiating with respect to time:

\[ L \frac{d^2 i(t)}{dt^2} + R \frac{di(t)}{dt} + \frac{1}{C} i(t) = \frac{dv_s(t)}{dt} \]
Second –Order Circuits

\[ \frac{d^2 i(t)}{dt^2} + \frac{R}{L} \frac{di(t)}{dt} + \frac{1}{LC} i(t) = \frac{1}{L} \frac{dv_s(t)}{dt} \]

Define:

\[ \alpha = \frac{R}{2L} \] Dampening coefficient

\[ \omega_0 = \frac{1}{\sqrt{LC}} \] Undamped resonant frequency

\[ f(t) = \frac{1}{L} \frac{dv_s(t)}{dt} \] Forcing function

\[ \frac{d^2 i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = f(t) \]
Solution of the Second-Order Equation

Particular solution

\[
\frac{d^2 i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = f(t)
\]

Complementary solution

\[
\frac{d^2 i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = 0
\]
Solution of the Complementary Equation

\[
\frac{d^2 i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = 0
\]

Try \( i_C(t) = Ke^{st} \):

\[
s^2 Ke^{st} + 2\alpha sKe^{st} + \omega_0^2 Ke^{st} = 0
\]

Factoring:

\[
(s^2 + 2\alpha s + \omega_0^2)Ke^{st} = 0
\]

Characteristic equation:

\[
s^2 + 2\alpha s + \omega_0^2 = 0
\]
Solution of the Complementary Equation

**Quadratic Equation:** \( ax^2 + bx + c = 0 \)

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

**Characteristic Equation:** \( s^2 + 2\alpha s + \omega_0^2 \)

\[
s = \frac{-2\alpha \pm \sqrt{(2\alpha)^2 - 4\omega_0^2}}{2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}
\]
Solution of the Complementary Equation

Roots of the characteristic equation:

\[ s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \]
\[ s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \]

\[ \zeta = \frac{\alpha}{\omega_0} \quad \text{Dampening ratio} \]
1. *Overdamped case* \((\zeta > 1)\). If \(\zeta > 1\) (or equivalently, if \(\alpha > \omega_0\)), the roots of the characteristic equation are real and distinct. Then the complementary solution is:

\[
i_c(t) = K_1e^{s_1t} + K_2e^{s_2t}
\]

In this case, we say that the circuit is *overdamped*. 
2. *Critically damped case* ($\zeta = 1$). If $\zeta = 1$ (or equivalently, if $\alpha = \omega_0$), the roots are real and equal. Then the complementary solution is

$$i_c(t) = K_1 e^{s_1 t} + K_2 t e^{s_1 t}$$

In this case, we say that the circuit is *critically damped.*
3. Underdamped case ($\zeta < 1$). Finally, if $\zeta < 1$ (or equivalently, if $\alpha < \omega_0$), the roots are complex. (By the term complex, we mean that the roots involve the square root of $-1$.) In other words, the roots are of the form:

$$s_1 = -\alpha + j\omega_n \text{ and } s_2 = -\alpha - j\omega_n$$

in which $j$ is the square root of $-1$ and the natural frequency is given by:

$$\omega_n = \sqrt{\omega_0^2 - \alpha^2}$$
In electrical engineering, we use \( j \) rather than \( i \) to stand for square root of \(-1\), because we use \( i \) for current. For complex roots, the complementary solution is of the form:

\[
i_c(t) = K_1 e^{-\alpha t} \cos(\omega_n t) + K_2 e^{-\alpha t} \sin(\omega_n t)
\]

In this case, we say that the circuit is \textit{underdamped}. 
Analysis of a Second-Order Circuit with a DC Source

Apply KVL around the loop:

\[ L \frac{di(t)}{dt} + Ri(t) + v_C(t) = V_S \]
Analysis of a Second-Order Circuit with a DC Source

\[ L \frac{di(t)}{dt} + Ri(t) + v_C(t) = V_S \]

Substitute \( i(t) = C \frac{dv_C(t)}{dt} \)

\[ LC \frac{d^2v_C(t)}{dt^2} + RC \frac{dv_C(t)}{dt} + v_C(t) = V_S \]

Divide by \( LC \):

\[ \frac{d^2v_C(t)}{dt^2} + \frac{R}{L} \frac{dv_C(t)}{dt} + \frac{1}{LC} v_C(t) = \frac{V_S}{LC} \]
Analysis of a Second-Order Circuit with a DC Source

To find the particular solution, we consider the steady state. For DC steady state, we can replace the inductors by a short circuit and the capacitor by an open circuit:

\[ V_{cp}(t) = V_s = 10V \]
Analysis of a Second-Order Circuit with a DC Source

To find the homogeneous solution, we consider three cases; R=300Ω, 200Ω and 100Ω. These values will correspond to over-damped, critically damped and under-damped cases.

\[ \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(10mH)(1\mu F)}} = 10^4 \text{ rads/sec} \]
Analysis of a Second-Order Circuit with a DC Source

Case I: \( R = 300 \, \Omega \)

\[
\alpha = \frac{R}{2L} = \frac{300 \, \Omega}{2(10 \, mH)} = 1.5 \times 10^4 \quad \zeta = \frac{\alpha}{\omega_0} = \frac{1.5 \times 10^4}{10^4} = 1.5 \rightarrow \text{Overdamped}
\]

\[
s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -1.5 \times 10^4 - \sqrt{(1.5 \times 10^4)^2 - (10^4)^2} = -2.618 \times 10^4
\]

\[
s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -0.382 \times 10^4
\]

Adding the particular and complementary solutions:

\[
v_C(t) = 10 + K_1 e^{s_1 t} + K_2 e^{s_2 t}
\]
Analysis of a Second-Order Circuit with a DC Source

To find $K_1$ and $K_2$ we use the initial conditions:

$$v_c(t) = 10 + K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

$$v_c(0) = 0 = 10 + K_1 e^0 + K_2 e^0 = 10 + K_1 + K_2$$

$$i(0) = 0$$

$$i(t) = C \frac{dv_c(t)}{dt} \quad i(0) = 0 \rightarrow \frac{dv_c(0)}{dt} = 0$$

$$\frac{dv_c(t)}{dt} = s_1 K_1 e^{s_1 t} + s_2 K_2 e^{s_2 t}$$

$$\frac{dv_c(0)}{dt} = 0 = s_1 K_1 e^0 + s_2 K_2 e^0 = s_1 K_1 + s_2 K_2$$
Analysis of a Second-Order Circuit with a DC Source

\[ 10 + K_1 + K_2 = 0 \]
\[ s_1 K_1 + s_2 K_2 = 0 \]

\[ K_1 + K_2 = -10 \]
\[ s_1 K_1 + s_2 K_2 = 0 \]

\[
K_1 = \begin{vmatrix}
-10 & 1 \\
0 & s_2 \\
1 & 1 \\
s_1 & s_2 \\
\end{vmatrix} = \frac{(-10)(s_2) - (0)(1)}{(1)(s_2) - (s_1)(1)} = \frac{-10s_2}{s_2 - s_1} = \frac{3.82 \times 10^4}{-0.3820 \times 10^4 - (-2.618 \times 10^4)} = 1.71
\]

\[
K_2 = \begin{vmatrix}
1 & -10 \\
s_1 & 0 \\
1 & 1 \\
s_1 & s_2 \\
\end{vmatrix} = \frac{(1)(0) - (s_1)(-10)}{(1)(s_2) - (s_1)(1)} = \frac{10s_1}{s_2 - s_1} = \frac{-26.18 \times 10^4}{-0.3820 \times 10^4 - (-2.618 \times 10^4)} = -11.7
\]

\[ v_c(t) = 10 + 1.708e^{s_1t} - 11.708e^{s_2t} \]
Analysis of a Second-Order Circuit with a DC Source

Case II :  \( R = 200\Omega \)

\[
\alpha = \frac{R}{2L} = \frac{200\Omega}{2(10mH)} = 10^4 \quad \zeta = \frac{\alpha}{\omega_0} = \frac{10^4}{10^4} = 1 \rightarrow \text{Critically damped}
\]

\[
s_1 = s_2 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -\alpha = -10^4
\]

Adding the particular and complementary solutions :

\[
v_C(t) = 10 + K_1e^{s_1t} + K_2te^{s_1t}
\]
Analysis of a Second-Order Circuit with a DC Source

To find $K_1$ and $K_2$ we use the initial conditions:

$$v_c(t) = 10 + K_1 e^{s_1 t} + K_2 t e^{s_1 t}$$
$$v_c(0) = 0 = 10 + K_1 e^0 + K_2 (0)e^0 = 10 + K_1$$

$$i(0) = 0$$
$$i(t) = C \frac{dv_c(t)}{dt} \quad i(0) = 0 \rightarrow \frac{dv_c(0)}{dt} = 0$$
$$\frac{dv_c(t)}{dt} = s_1 K_1 e^{s_1 t} + s_1 K_2 t e^{s_1 t} + K_2 e^{s_1 t}$$
$$\frac{dv_c(0)}{dt} = 0 = s_1 K_1 e^0 + s_1 K_2 (0)e^0 + K_2 e^0 = s_1 K_1 + K_2$$
Analysis of a Second-Order Circuit with a DC Source

\[ 10 + K_1 = 0 \quad \rightarrow \quad K_1 = -10 \]

\[ s_1 K_1 + K_2 = 0 \quad \rightarrow \quad K_2 = -s_1 K_1 = 10 s_1 = 10(-10^4) = -10^5 \]

\[ v_c(t) = 10 - 10 e^{s_1 t} - 10^5 t e^{s_2 t} \]
Analysis of a Second-Order Circuit with a DC Source

Case III :  $R = 100\Omega$

\[
\alpha = \frac{R}{2L} = \frac{100\Omega}{2(10mH)} = 0.5 \times 10^4 \quad \zeta = \frac{\alpha}{\omega_0} = \frac{0.5 \times 10^4}{10^4} = 0.5 \rightarrow \text{Underdamped}
\]

\[
\omega_n = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{(10^4)^2 - (0.5 \times 10^4)^2} = 8660
\]

Adding the particular and complementary solutions:

\[
v_C(t) = 10 + K_1 e^{-\alpha t} \cos(\omega_n t) + K_2 e^{-\alpha t} \sin(\omega_n t)
\]

\[
\frac{dv_C(t)}{dt} = -\alpha K_1 e^{-\alpha t} \cos(\omega_n t) - K_1 e^{-\alpha t} \omega_n \sin(\omega_n t) - \alpha K_2 e^{-\alpha t} \sin(\omega_n t) + K_2 e^{-\alpha t} \omega_n \cos(\omega_n t)
\]
Analysis of a Second-Order Circuit with a DC Source

To find $K_1$ and $K_2$ we use the initial conditions:

\[ v_c(0) = 0 \]
\[ \frac{dv_c(0)}{dt} = 0 \]

\[ 10 + K_1 = 0 \quad \rightarrow \quad K_1 = -10 \]
\[ -\alpha K_1 + \omega_n K_2 = 0 \quad \rightarrow \quad K_2 = \frac{\alpha}{\omega_n} K_1 = \frac{5000}{8660} (-10) = -5.774 \]

\[ v_c(t) = 10 - 10e^{-\alpha t} \cos(\omega_n t) - 5.774e^{-\alpha t} \sin(\omega_n t) \]
Figure 4.23  Solution for $R = 300 \, \Omega$.

Overdamped
Figure 4.24 Solution for $R = 200 \, \Omega$.

Critically damped
Figure 4.25 Solution for $R = 100 \, \Omega$.

Underdamped
Figure 4.26 Solutions for all three resistances.
Normalized Step Response of a Second Order System

$$u(t) = 0 \quad t < 0$$

$$u(t) = 1 \quad t > 0$$
Normalized Step Response of a Second Order System

Example: Closing a switch at $t=0$ to apply a DC voltage $A$

\[ v(t) = Au(t) \]
What value of $\zeta$ should be used to get to the steady state position quickly without overshooting?

Figure 4.29 Normalized step responses for second-order systems described by Equation 4.99 with damping ratios of $\zeta = 0.1$, 0.5, 1, 2, and 3. The initial conditions are assumed to be $x(0) = 0$ and $x'(0) = 0$. 
Circuits with Parallel L and C

We can replace the circuit with it’s Norton equivalent and then analyze the circuit by writing KCL at the top node:

\[ C \frac{dv(t)}{dt} + \frac{1}{R} v(t) + \frac{1}{L} \int_{0}^{t} v(t) dt + i_L(0) = i_n(t) \]
Circuits with Parallel L and C

\[ C \frac{dv(t)}{dt} + \frac{1}{R} v(t) + \frac{1}{L} \int_0^t v(t) dt + i_L(0) = i_n(t) \]

differentiating:

\[ C \frac{d^2v(t)}{dt} + \frac{1}{R} \frac{dv(t)}{dt} + \frac{1}{L} v(t) = \frac{di_n(t)}{dt} \]

\[ \frac{d^2v(t)}{dt} + \frac{1}{RC} \frac{dv(t)}{dt} + \frac{1}{LC} v(t) = \frac{1}{C} \frac{di_n(t)}{dt} \]
Circuits with Parallel L and C

\[
\frac{d^2 v(t)}{dt} + \frac{1}{RC} \frac{dv(t)}{dt} + \frac{1}{LC} v(t) = \frac{1}{C} \frac{di_n(t)}{dt}
\]

Dampening coefficient \( \alpha = \frac{1}{2RC} \)

Undamped resonant frequency \( \omega_0 = \frac{1}{\sqrt{LC}} \)

Forcing function \( f(t) = \frac{1}{C} \frac{di_n(t)}{dt} \)

\[
\frac{d^2 v(t)}{dt} + 2\alpha \frac{dv(t)}{dt} + \omega_0^2 v(t) = f(t)
\]
Circuits with Parallel L and C

\[
\frac{d^2v(t)}{dt} + 2\alpha \frac{dv(t)}{dt} + \omega_0^2 v(t) = f(t)
\]

This is the same equation as we found for the series LC circuit with the following changes for \(\alpha\):

- **Parallel circuit** \(\alpha = \frac{1}{2RC}\)
- **Series circuit** \(\alpha = \frac{R}{2L}\)
Example Parallel LC Circuit: RF-ID Tag

Circuits with Parallel L and C

Find $v(t)$ for $R=25\,\Omega$, $50\,\Omega$ and $250\,\Omega$
Circuits with Parallel L and C

Find $v(t)$ for $R=25\Omega$, $50\Omega$ and $250\Omega$

\[ \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(1\times10^{-3}\text{H})(1\times10^{-7}\text{F})}} = 1\times10^5 \text{ rad/sec} \]

\[ \alpha = \frac{1}{2RC} \]
Circuits with Parallel L and C

To find the particular solution $v_p(t)$ (steady state response) replace C with an open circuit and L with a short circuit.
Circuits with Parallel L and C

Case I: \( R = 25\Omega \)

\[
\alpha = \frac{1}{2RC} = \frac{1}{2(25\Omega)(1\times10^{-7} F)} = 2\times10^5 \text{ s}^{-1} \\
\zeta = \frac{\alpha}{\omega_0} = \frac{2\times10^5 \text{ s}^{-1}}{10^5 \text{ rad} / \text{s}} = 2 \rightarrow \text{Overdamped}
\]

\[
s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -2\times10^5 + \sqrt{(2\times10^5)^2 - (10^5)^2} = -0.268\times10^5
\]

\[
s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -2\times10^5 - \sqrt{(2\times10^5)^2 - (10^5)^2} = -3.73\times10^5
\]

Adding the particular and complementary solutions:

\[
\begin{align*}
\nu_C(t) &= K_1 e^{s_1 t} + K_2 e^{s_2 t} \\
\nu_P(t) &= 0 \\
\nu_T(t) &= K_1 e^{s_1 t} + K_2 e^{s_2 t}
\end{align*}
\]
Circuits with Parallel L and C

**Initial conditions:**

\[ v(0-) = 0 \rightarrow v(0+) = 0 \]
\[ i_L(0-) = 0 \rightarrow i_L(0+) = 0 \]

Since the voltage across a capacitor and the current through an inductor cannot change instantaneously.

**KCL at \( t = 0^+ \)**

\[ 0.1 = \frac{v(0+)}{R} + i_L(0+) + C \frac{dv(0+)}{dt} \]
\[ 0.1 = C \frac{dv(0+)}{dt} \rightarrow \frac{dv(0+)}{dt} = \frac{0.1}{C} = \frac{0.1}{0.1 \times 10^{-6}} = 10^6 \text{ v/s} \]

\[ v(0+) = K_1 e^0 + K_2 e^0 \rightarrow K_1 + K_2 = 0 \]
\[ \frac{dv(0+)}{dt} = K_1 s_1 e^0 + K_2 s_2 e^0 = K_1 s_1 + K_2 s_2 = 10^6 \]
Circuits with Parallel L and C

\[ K_1 + K_2 = 0 \]
\[ s_1 K_1 + s_2 K_2 = 10^6 \]

\[
K_1 = \begin{vmatrix} 0 & 1 \\ 10^6 & s_2 \\ s_1 & s_2 \end{vmatrix} = \frac{(0)(s_2) - (10^6)(1)}{(1)(s_2) - (s_1)(1)} = \frac{-10^6}{s_2 - s_1} = \frac{-10^6}{-0.268 \times 10^5 - (-3.73 \times 10^5)} = 2.89
\]

\[ K_2 = -K_1 = -2.89 \]

\[ v(t) = 2.89 e^{-0.268 \times 10^5 t} - 2.89 e^{-3.73 \times 10^5 t} = 2.89(e^{-0.268 \times 10^5 t} - e^{-3.73 \times 10^5 t}) \]
Circuits with Parallel L and C

Case II: \( R = 50\Omega \)

\[
\alpha = \frac{1}{2RC} = \frac{1}{2(50\Omega)(1x10^{-7} F)} = 1x10^5 \text{ s}^{-1}
\]

\[
\zeta = \frac{\alpha}{\omega_0} = \frac{1x10^5 \text{ s}^{-1}}{10^5 \text{ rad} / \text{s}} = 1 \rightarrow \text{Critically damped}
\]

\( s_1 = -\alpha = -1x10^5 \)

Adding the particular and complementary solutions:

\( v_C(t) = K_1 e^{s_1 t} + K_2 te^{s_1 t} \)

\( v_P(t) = 0 \)

\( v_T(t) = K_1 e^{s_1 t} + K_2 te^{s_1 t} \)
Circuits with Parallel L and C

\[ v(t) = K_1 e^{s_1 t} + K_2 te^{s_1 t} \]

\[ v(0) = K_1 e^0 + K_2 (0)e^0 = K_1 = 0 \]

\[ v(t) = K_2 te^{s_1 t} \]

\[ \frac{dv(t)}{dt} = K_2 e^{s_1 t} + s_1 K_2 te^{s_1 t} \]

\[ \frac{dv(0^+)}{dt} = K_2 e^0 = 10^6 \rightarrow K_2 = 10^6 \]

\[ v(t) = 10^6 te^{-10^5 t} \]
Circuits with Parallel L and C

Case III:  \( R = 250\Omega \)

\[
\alpha = \frac{1}{2RC} = \frac{1}{2(250\Omega)(1 \times 10^{-7} F)} = 0.2 \times 10^5 \text{ s}^{-1}
\]

\[
\zeta = \frac{\alpha}{\omega_0} = \frac{0.2 \times 10^5 \text{ s}^{-1}}{10^5 \text{ rad/s}} = 0.2 \rightarrow \text{Underdamped}
\]

\[
\omega_n = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{(1 \times 10^5)^2 - (0.2 \times 10^5)^2} = 0.98 \times 10^5
\]

\[
s_1 = -\alpha + j\omega_n = -0.2 \times 10^5 + j0.98 \times 10^5
\]

\[
s_2 = -\alpha - j\omega_n = -0.2 \times 10^5 - j0.98 \times 10^5
\]

Adding the particular and complementary solutions:

\[
v_C(t) = K_1 e^{-\alpha t} \cos(\omega_n t) + K_2 e^{-\alpha t} \sin(\omega_n t)
\]

\[
v_F(t) = 0
\]

\[
v_T(t) = K_1 e^{-\alpha t} \cos(\omega_n t) + K_2 e^{-\alpha t} \sin(\omega_n t)
\]
Circuits with Parallel L and C

\[ v(t) = K_1 e^{-\alpha t} \cos(\omega_n t) + K_2 e^{-\alpha t} \sin(\omega_n t) \]

\[ \frac{dv(t)}{dt} = -\alpha K_1 e^{-\alpha t} \cos(\omega_n t) - \omega_n K_1 e^{-\alpha t} \sin(\omega_n t) - \alpha K_2 e^{-\alpha t} \sin(\omega_n t) + \omega_n K_2 e^{-\alpha t} \cos(\omega_n t) \]

\[ v(0) = 0 = K_1 e^{-\alpha t} \rightarrow K_1 = 0 \]

\[ \frac{dv(0)}{dt} = 10^6 = -\alpha K_1 + \omega_n K_2 = \omega_n K_2 \rightarrow K_2 = \frac{10^6}{0.98 \times 10^5} = 10.2 \]

\[ v(t) = (10.2) e^{-0.2 \times 10^5 t} \sin(0.98 \times 10^5 t) \]