EE 101 Midterm 2 Review
Voltage-Amplifier Model

The input resistance $R_i$ is the equivalent resistance seen when looking into the input terminals of the amplifier. $R_o$ is the output resistance. It causes the output voltage to decrease as the load resistance becomes smaller. $A_{\text{oc}}$ is the open circuit voltage gain.
Voltage-Amplifier Model

Thevenin equivalents

Negative Feedback
IDEAL OPERATIONAL AMPLIFIERS

Figure 14.1 Circuit symbol for the op amp.

Figure 14.2 Equivalent circuit for the ideal op amp. The open-loop gain $A_{OL}$ is very large (approaching infinity).
SUMMING-POINT CONSTRAINT

Operational amplifiers are almost always used with negative feedback, in which part of the output signal is returned to the input in opposition to the source signal.

In a negative feedback system, the ideal op-amp output voltage attains the value needed to force the differential input voltage and input current to zero. We call this fact the **summing-point constraint**.
Ideal op-amp circuits are analyzed by the following steps:

1. Verify that *negative* feedback is present.

2. Assume that the differential input voltage and the input current of the op amp are forced to zero. (This is the summing-point constraint.)

3. Apply standard circuit-analysis principles, such as Kirchhoff’s laws and Ohm’s law, to solve for the quantities of interest.
The Basic Inverter

![The Basic Inverter Diagram](image)

Figure 14.4 The inverting amplifier.

Applying the Summing Point Constraint

![Applying the Summing Point Constraint Diagram](image)

Figure 14.5 We make use of the summing-point constraint in the analysis of the inverting amplifier.
Applying the Summing Point Constraint

\[ i_1 = \frac{v_{in} - 0}{R_1} = \frac{v_{in}}{R_1} \]
\[ i_2 = \frac{0 - v_{out}}{R_2} = -\frac{v_{out}}{R_2} \]
\[ i_1 = i_2 \rightarrow \frac{v_{in}}{R_1} = -\frac{v_{out}}{R_2} \]
\[ \frac{v_{out}}{v_{in}} = -\frac{R_2}{R_1} \]

Summing Amplifier

Figure 14.7 Summing amplifier. See Exercise 14.1.
**Summing Amplifier**

\[
\begin{align*}
i_A &= \frac{V_A}{R_A} \\
i_B &= \frac{V_B}{R_B} \\
i_F &= i_A + i_B = \frac{V_A}{R_A} + \frac{V_B}{R_B} \\
i_F &= 0 - v_o \frac{R_F}{R_F} = -v_o \frac{V_A}{R_A} + \frac{V_B}{R_B} \Rightarrow v_o = -R_F \left( \frac{V_A}{R_A} + \frac{V_B}{R_B} \right)
\end{align*}
\]

**Summing Amplifier**

\[
\begin{align*}
i_A &= \frac{V_A}{R_A} \\
i_B &= \frac{V_B}{R_B} \\
i_{out} &= \frac{0 - v_{out}}{R_F} = -v_{out} \frac{R_F}{R_F} \\
i_{out} &= i_A + i_B \rightarrow -v_{out} \frac{v_A}{R_A} + \frac{v_B}{R_B} \\
v_{out} &= -R_F \left( \frac{v_A}{R_A} + \frac{v_B}{R_B} \right)
\end{align*}
\]
Non-inverting Amplifier

\[ i_i \rightarrow v_1 = v_{in} \]

\[ v_i = \frac{R_1}{R_1 + R_2} v_o \rightarrow v_o = \frac{R_1 + R_2}{R_1} v_{in} = \left(1 + \frac{R_2}{R_1}\right) v_{in} \]

\[ A_v = \frac{v_o}{v_{in}} = \left(1 + \frac{R_2}{R_1}\right) \]

Voltage Follower

\[ A_v = \frac{v_o}{v_{in}} = 1 + \frac{R_2}{R_1} = 1 + \frac{0}{\infty} = 1 \]
\[ i_{in} = \frac{v_{in}}{R} \quad v_c = \frac{q}{C} = \frac{1}{C} \int_{0}^{t} i_{in} \, dt = \frac{1}{RC} \int_{0}^{t} v_{in} \, dt \]

\[-v_o - v_c = 0 \quad \Rightarrow \quad v_o = -v_c = -\frac{1}{RC} \int_{0}^{t} v_{in} \, dt \]

\[ v_o(t) = -\frac{1}{RC} \int_{0}^{t} v_{in}(t) \, dt \]
Differentiator Circuit

\[ i_{in} = \frac{dq}{dt} = C \frac{dv_{in}}{dt} = \frac{0 - v_o}{R} \Rightarrow v_o = -RC \frac{dv_{in}}{dt} \]

Differentiator Circuit

\[ v_o(t) = -RC \frac{dv_{in}}{dt} \]
Lecture 11
Inductance and Capacitance

Energy is stored in the electric field that exists between the plates when the capacitor is charged.

\[ C = \frac{q}{v} \]
Capacitance

\[ C = \frac{q}{v} \quad v = \frac{q}{C} \]

\[ q(t) = \int_{t_0}^{t} i(t) dt + q(t_0) \]

\[ v(t) = \frac{1}{C} \int_{t_0}^{t} i(t) dt + v(t_0) \]

Inductance

\[ v(t) = LI \frac{di}{dt} \]

\[ i(t) = \frac{1}{L} \int_{t_0}^{t} v(t) dt + i(t_0) \]

\[ p(t) = v(t)i(t) = LI(t) \frac{di}{dt} \]

\[ w(t) = \int_{0}^{t} p(t) dt = \int_{0}^{t} LI \frac{di}{dt} dt = \int_{0}^{t} Lidi = \frac{1}{2} LI^2(t) \]
Lecture 12
First Order Transient Response

Discharge of a Capacitance through a Resistance

KCL at the top node of the circuit:

\[ C \frac{dq}{dt} = C \frac{dv}{dt} \]
\[ i_R = \frac{v_C(t)}{R} \]

Charge conservation:

\[ q = C v \rightarrow i_C = \frac{dq}{dt} = C \frac{dv}{dt} \]
\[ i_R = \frac{v_C(t)}{R} \]

\[ C \frac{dv_C(t)}{dt} + \frac{v_C(t)}{R} = 0 \]
\[ RC \frac{dv_C(t)}{dt} + v_C(t) = 0 \]
Discharge of a Capacitance through a Resistance

$$RC \frac{dv_C(t)}{dt} + v_C(t) = 0 \quad \Rightarrow \quad \frac{dv_C(t)}{dt} = -\frac{1}{RC} v_C(t)$$

We need a function $v_c(t)$ that has the same form as it’s derivative.

$$v_C(t) = Ke^{st}$$

Substituting this in for $v_c(t)$

$$RCKse^{st} + Ke^{st} = 0$$

Discharge of a Capacitance through a Resistance

Solving for $s$: \[ s = \frac{-1}{RC} \]

Substituting into $v_c(t)$: \[ v_c(t) = Ke^{-t/RC} \]

Initial Condition: \[ v_c(0 +) = V_i \]

Full Solution: \[ v_c(t) = V_i e^{-t/RC} \]
Discharge of a Capacitance through a Resistance

\[ v_C(t) = Ke^{-t/RC} \]

To find the unknown constant \( K \), we need to use the boundary conditions at \( t=0 \). At \( t=0 \) the capacitor is initially charged to a voltage \( V_i \) and then discharges through the resistor.

\[ v_C(0+) = V_i \quad \quad v_C(t) = V_i e^{-t/RC} \]

Charging a Capacitance from a DC Source through a Resistance
Charging a Capacitance from a DC Source through a Resistance

KCL at the node that joins the resistor and the capacitor

Current into the capacitor: $C \frac{dv_C}{dt}$

Current through the resistor: $\frac{v_C(t) - V_S}{R}$

$$C \frac{dv_C(t)}{dt} + \frac{v_C(t) - V_S}{R} = 0$$

Rearranging:

$$RC \frac{dv_C(t)}{dt} + v_C(t) = V_S$$

This is a linear first-order differential equation with constant coefficients.
Charging a Capacitance from a DC Source through a Resistance

The boundary conditions are given by the fact that the voltage across the capacitance cannot change instantaneously:

\[ v_C(0+) = v_C(0-) = 0 \]

Try the solution:

\[ v_C(t) = K_1 + K_2 e^{st} \]

Substituting into the differential equation:

\[ RC \frac{dv_C(t)}{dt} + v_C(t) = V_S \]

Gives:

\[ (1 + RCs)K_2 e^{st} + K_1 = V_S \]
Charging a Capacitance from a DC Source through a Resistance

\[(1 + RC_s)K_2e^{st} + K_1 = V_S\]

For equality, the coefficient of \(e^{st}\) must be zero:

\[1 + RC_s = 0 \rightarrow s = \frac{-1}{RC}\]

Which gives \(K_1 = V_S\)

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Charging a Capacitance from a DC Source through a Resistance

Substituting in for \(K_1\) and \(s\):

\[v_C(t) = K_1 + K_2e^{st} = V_S + K_2e^{-t/RC}\]

Evaluating at \(t=0\) and remembering that \(v_C(0+) = 0\)

\[v_C(0+) = V_S + K_2e^0 = V_S + K_2 = 0 \rightarrow K_2 = -V_S\]

Substituting in for \(K_2\) gives:

\[v_C(t) = K_1 + K_2e^{st} = V_S - V_S e^{-t/RC}\]
DC Steady State

\[ i_C(t) = C \frac{dv_C(t)}{dt} \]

In steady state, the voltage is constant, so the current through the capacitor is zero, so it behaves as an open circuit.

DC Steady State

\[ v_L(t) = L \frac{di_L(t)}{dt} \]

In steady state, the current is constant, so the voltage across and inductor is zero, so it behaves as a short circuit.
DC Steady State

The steps in determining the forced response for \textit{RLC} circuits with dc sources are:

1. Replace capacitances with open circuits.
2. Replace inductances with short circuits.
3. Solve the remaining circuit.

Lecture 13
RC/RL Circuits, Time Dependent Op Amp Circuits
RL Circuits

The steps involved in solving simple circuits containing dc sources, resistances, and one energy-storage element (inductance or capacitance) are:

1. Apply Kirchhoff’s current and voltage laws to write the circuit equation.

2. If the equation contains integrals, differentiate each term in the equation to produce a pure differential equation.

3. Assume a solution of the form $K_1 + K_2e^{st}$. 
4. Substitute the solution into the differential equation to determine the values of $K_1$ and $s$. (Alternatively, we can determine $K_1$ by solving the circuit in steady state)

5. Use the initial conditions to determine the value of $K_2$.

6. Write the final solution.

**RL Transient Analysis**

Find $i(t)$ and the voltage $v(t)$

$i(t) = 0$ for $t < 0$ since the switch is open prior to $t = 0$

Apply KVL around the loop:

$$-V_S + i(t)R + v(t) = 0$$
RL Transient Analysis

\[-V_S + i(t)R + v(t) = 0\]

\[i(t)R + L \frac{di(t)}{dt} = V_S\]

RL Transient Analysis

\[i(t)R + L \frac{di(t)}{dt} = V_S\]

Try \[i(t) = K_1 + K_2 e^{st}\]

\[RK_1 + (RK_2 + sLK_2) e^{st} = V_S\]

\[RK_1 = V_S \rightarrow K_1 = \frac{V_S}{R} = \frac{100V}{50\Omega} = 2A\]

\[RK_2 + sLK_2 = 0 \rightarrow s = -\frac{R}{L}\]
**RL Transient Analysis**

\[ i(t) = 2 + K_2 e^{-tR/L} \]

\[ i(0^+) = 0 = 2 + K_2 e^0 = 2 + K_2 \rightarrow K_2 = -2 \]

\[ i(t) = 2 - 2e^{-tR/L} \]

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**RC and RL Circuits with General Sources**

First order differential equation with constant coefficients

\[ L \frac{di(t)}{dt} + Ri(t) = v_i(t) \]

\[ \frac{L}{R} \frac{di(t)}{dt} + i(t) = \frac{v_i(t)}{R} \]

\[ \tau \frac{dx(t)}{dt} + x(t) = f(t) \]
RC and RL Circuits with General Sources

The general solution consists of two parts.

The particular solution (also called the forced response) is any expression that satisfies the equation.

\[ \tau \frac{dx(t)}{dt} + x(t) = f(t) \]

In order to have a solution that satisfies the initial conditions, we must add the complementary solution to the particular solution.
The homogeneous equation is obtained by setting the forcing function to zero.

\[ \tau \frac{dx(t)}{dt} + x(t) = 0 \]

The complementary solution (also called the natural response) is obtained by solving the homogeneous equation.

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**Step-by-Step Solution**

Circuits containing a resistance, a source, and an inductance (or a capacitance)

1. Write the circuit equation and reduce it to a first-order differential equation.
2. Find a particular solution. The details of this step depend on the form of the forcing function.

3. Obtain the complete solution by adding the particular solution to the complementary solution $x_c = Ke^{-t/\tau}$ which contains the arbitrary constant $K$.

4. Use initial conditions to find the value of $K$. 

Lecture 14
Second Order Transient Response
Second -Order Circuits

\[ L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int_0^t i(t) \, dt + v_c(0) = v_s(t) \]

Differentiating with respect to time:

\[ L \frac{d^2i(t)}{dt^2} + R \frac{di(t)}{dt} + \frac{1}{C} i(t) = \frac{dv_s(t)}{dt} \]

Define:

- \( \alpha = \frac{R}{2L} \) \hspace{.5cm} \text{Dampening coefficient}
- \( \omega_0 = \frac{1}{\sqrt{LC}} \) \hspace{.5cm} \text{Undamped resonant frequency}
- \( f(t) = \frac{1}{L} \frac{dv_s(t)}{dt} \) \hspace{.5cm} \text{Forcing function}

\[ \frac{d^2i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = f(t) \]
Solution of the Second-Order Equation

**Particular solution**

\[
\frac{d^2 i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = f(t)
\]

**Complementary solution**

\[
\frac{d^2 i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = 0
\]

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Solution of the Complementary Equation

\[
\frac{d^2 i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = 0
\]

Try \( i_C(t) = Ke^{st} \):

\[
s^2 Ke^{st} + 2s\alpha Ke^{st} + \omega_0^2 Ke^{st} = 0
\]

Factoring:

\[
(s^2 + 2\alpha s + \omega_0^2)Ke^{st} = 0
\]

Characteristic equation:

\[
s^2 + 2\alpha s + \omega_0^2 = 0
\]
Solution of the Complementary Equation

**Quadratic Equation:** \( ax^2 + bx + c = 0 \)

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

**Characteristic Equation:** \( s^2 + 2\alpha s + \omega_0^2 \)

\[
s = \frac{-2\alpha \pm \sqrt{(2\alpha)^2 - 4\omega_0^2}}{2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}
\]

Solution of the Complementary Equation

Roots of the characteristic equation:

\[
s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}
\]

\[
s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}
\]

\[
\zeta = \frac{\alpha}{\omega_0} \quad \text{Dampening ratio}
\]
1. **Overdamped case** ($\zeta > 1$). If $\zeta > 1$ (or equivalently, if $\alpha > \omega_0$), the roots of the characteristic equation are real and distinct. Then the complementary solution is:

$$i_c(t) = K_1e^{s_1t} + K_2e^{s_2t}$$

In this case, we say that the circuit is **overdamped**.

2. **Critically damped case** ($\zeta = 1$). If $\zeta = 1$ (or equivalently, if $\alpha = \omega_0$), the roots are real and equal. Then the complementary solution is

$$i_c(t) = K_1e^{s_1t} + K_2te^{s_1t}$$

In this case, we say that the circuit is **critically damped**.
3. **Underdamped case** \((\zeta < 1)\). Finally, if \(\zeta < 1\) (or equivalently, if \(\alpha < \omega_0\)), the roots are complex. (By the term *complex*, we mean that the roots involve the square root of \(-1\).) In other words, the roots are of the form:

\[
s_1 = -\alpha + j\omega_n \quad \text{and} \quad s_2 = -\alpha - j\omega_n
\]

in which \(j\) is the square root of \(-1\) and the **natural frequency** is given by:

\[
\omega_n = \sqrt{\omega_0^2 - \alpha^2}
\]

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**Circuits with Parallel L and C**

We can replace the circuit with its Norton equivalent and then analyze the circuit by writing KCL at the top node:

\[
C \frac{dv(t)}{dt} + \frac{1}{R} v(t) + \frac{1}{L} \int_0^t v(t) dt + i_L(0) = i_n(t)
\]
Circuits with Parallel L and C

\[ C \frac{dv(t)}{dt} + \frac{1}{R} v(t) + \frac{1}{L} \int _0 ^t v(t) dt + i_L (0) = i_n (t) \]

differentiating:

\[ C \frac{d^2 v(t)}{dt^2} + \frac{1}{R} \frac{dv(t)}{dt} + \frac{1}{L} v(t) = \frac{di_n (t)}{dt} \]

\[ \frac{d^2 v(t)}{dt^2} + \frac{1}{RC} \frac{dv(t)}{dt} + \frac{1}{LC} v(t) = \frac{1}{C} \frac{di_n (t)}{dt} \]

Circuits with Parallel L and C

\[ \frac{d^2 v(t)}{dt^2} + \frac{1}{RC} \frac{dv(t)}{dt} + \frac{1}{LC} v(t) = \frac{1}{C} \frac{di_n (t)}{dt} \]

Dampening coefficient \( \alpha = \frac{1}{2RC} \)

Undamped resonant frequency \( \omega_0 = \frac{1}{\sqrt{LC}} \)

Forcing function \( f(t) = \frac{1}{C} \frac{di_n (t)}{dt} \)

\[ \frac{d^2 v(t)}{dt^2} + 2\alpha \frac{dv(t)}{dt} + \omega_0 ^2 v(t) = f(t) \]
Circuits with Parallel L and C

\[
\frac{d^2 v(t)}{dt^2} + 2\alpha \frac{dv(t)}{dt} + \omega_0^2 v(t) = f(t)
\]

This is the same equation as we found for the series LC circuit with the following changes for \( \alpha \):

**Parallel circuit** \( \alpha = \frac{1}{2RC} \)

**Series circuit** \( \alpha = \frac{R}{2L} \)