Reference Directions

Energy supplied by the element

“uphill: battery”

Energy absorbed by the element

“downhill: resistor”

Figure 1.10 Energy is transferred when charge flows through an element having a voltage across it.
Reference Directions

Figure 1.12 The voltage \( v_{ab} \) has a reference polarity that is positive at point \( a \) and negative at point \( b \).

Power and Energy

\[
p(t) = v(t)i(t) \quad \text{Watts}
\]

\[
w = \int_{t_1}^{t_2} p(t)dt \quad \text{Joules}
\]
Reference Directions

If the current flows opposite to the passive configuration, the power is given by $p = -vi$.

Resistors and Ohm’s Law

The units of resistance are Volts/Amp which are called “ohms”. The symbol for ohms is omega: $\Omega$. 

$\nu = iR$

$\nu_{ab} = i_{ab}R$
Resistance Related to Physical Parameters

\[ R = \frac{\rho L}{A} \]

\( \rho \) is the resistivity of the material used to fabricate the resistor. The units of resistivity are ohm-meters \((\Omega \cdot m)\).

Power dissipation in a resistor

\[ p = vi = Ri^2 = \frac{v^2}{R} \]
Reference Directions

Figure 1.23 In applying KVL to a loop, voltages are added or subtracted depending on their reference polarities relative to the direction of travel around the loop.

Circuit Laws, Voltage & Current Dividers
Kirchhoff’s Current Law

• The net current entering a node is zero.

• Alternatively, the sum of the currents entering a node equals the sum of the currents leaving a node.

Kirchhoff’s Voltage Law

The algebraic sum of the voltages equals zero for any closed path (loop) in an electrical circuit.
Series Connection

\[
i = i_R + i_B + i_C = \frac{v}{R_1} + \frac{v}{R_2} + \frac{v}{R_3} = \frac{iR_1 + iR_2 + iR_3}{R_1 + R_2 + R_3} = iR_{eq}
\]

Figure 2.1 Series resistances can be combined into an equivalent resistance.
Parallel Connection

\[ i = i_1 + i_2 + i_3 = \frac{v}{R_1} + \frac{v}{R_2} + \frac{v}{R_3} = \frac{v}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} = \frac{v}{R_{eq}} \]

Figure 2.2 Parallel resistances can be combined into an equivalent resistance.

Circuit Analysis using Series/Parallel Equivalents

1. Begin by locating a combination of resistances that are in series or parallel. Often the place to start is farthest from the source.

2. Redraw the circuit with the equivalent resistance for the combination found in step 1.
3. Repeat steps 1 and 2 until the circuit is reduced as far as possible. Often (but not always) we end up with a single source and a single resistance.

4. Solve for the currents and voltages in the final equivalent circuit.

Voltage Division

\[ v_1 = R_1i = \frac{R_1}{R_1 + R_2 + R_3} v_{\text{total}} \]

\[ v_2 = R_2i = \frac{R_2}{R_1 + R_2 + R_3} v_{\text{total}} \]

Of the total voltage, the fraction that appears across a given resistance in a series circuit is the ratio of the given resistance to the total series resistance.
Current Division

\[ i_1 = \frac{v}{R_1} = \frac{R_2}{R_1 + R_2} i_{\text{total}} \]

\[ i_2 = \frac{v}{R_2} = \frac{R_1}{R_1 + R_2} i_{\text{total}} \]

For two resistances in parallel, the fraction of the total current flowing in a resistance is the ratio of the other resistance to the sum of the two resistances.

Node/Loop Analysis
Node Voltage Analysis

1. Select a reference node and assign variables for the unknown node voltages. If the reference node is chosen at one end of an independent voltage source, one node voltage is known at the start, and fewer need to be computed.
Node-Voltage Analysis

2. Write network equations. First, use KCL to write current equations for nodes and supernodes. Write as many current equations as you can without using all of the nodes. Then if you do not have enough equations because of voltage sources connected between nodes, use KVL to write additional equations.

Node-Voltage Analysis

3. If the circuit contains dependent sources, find expressions for the controlling variables in terms of the node voltages. Substitute into the network equations, and obtain equations having only the node voltages as unknowns.
Node-Voltage Analysis

4. Put the equations into standard form and solve for the node voltages.

5. Use the values found for the node voltages to calculate any other currents or voltages of interest.

Node/Loop Analysis
Mesh Current Analysis

Choosing the Mesh Currents

When several mesh currents flow through one element, we consider the current in that element to be the algebraic sum of the mesh currents.

Sometimes it is said that the mesh currents are defined by “soaping the window panes.”
Choosing the Mesh Currents

Mesh Current Analysis

Solve for the mesh currents:

\[ 20i_1 + 10(i_1 - i_2) - 150 = 0 \]
\[ 15i_2 + 100 + 10(i_2 - i_1) = 0 \]
Mesh Current Analysis

\[ 20i_1 + 10(i_1 - i_2) - 150 = 0 \]
\[ 15i_2 + 100 + 10(i_2 - i_1) = 0 \]

Putting the equations into the standard format:

\[ 30i_1 - 10i_2 = 150 \]
\[ -10i_1 + 25i_2 = -100 \]

Super-Mesh

![Figure 2.36](image) A circuit with a current source common to two meshes.
Combine meshes 1 and 2 into a **supermesh**. In other words, we write a KVL equation around the periphery of meshes 1 and 2 combined.

\[
 i_1 + 2(i_1 - i_3) + 4(i_2 - i_3) + 10 = 0
\]

Mesh 3:

\[
 3i_3 + 4(i_3 - i_2) + 2(i_3 - i_1) = 0
\]

\[
 i_2 - i_1 = 5
\]

---

**Mesh-Current Analysis**

1. If necessary, redraw the network without crossing conductors or elements. Then define the mesh currents flowing around each of the open areas defined by the network. For consistency, we usually select a clockwise direction for each of the mesh currents, but this is not a requirement.
2. Write network equations, stopping after the number of equations is equal to the number of mesh currents. First, use KVL to write voltage equations for meshes that do not contain current sources. Next, if any current sources are present, write expressions for their currents in terms of the mesh currents. Finally, if a current source is common to two meshes, write a KVL equation for the supermesh.

3. If the circuit contains dependent sources, find expressions for the controlling variables in terms of the mesh currents. Substitute into the network equations, and obtain equations having only the mesh currents as unknowns.
Mesh-Current Analysis

4. Put the equations into standard form. Solve for the mesh currents by use of determinants or other means.

5. Use the values found for the mesh currents to calculate any other currents or voltages of interest.

Thévenin Equivalent Circuits
Thévenin Equivalent Circuits

Figure 2.40 A two-terminal circuit consisting of resistances and sources can be replaced by a Thévenin equivalent circuit.

Figure 2.41 Thévenin equivalent circuit with open-circuited terminals. The open-circuit voltage $v_{oc}$ is equal to the Thévenin voltage $V_t$.

$$V_t = v_{oc}$$
Thévenin Equivalent Circuits

\[ i_{sc} = \frac{V_t}{R_t} \]

*Figure 2.42* Thévenin equivalent circuit with short-circuited terminals. The short-circuit current is \( i_{sc} = V_t / R_t \).

Thévenin Equivalent Circuits

\[ R_t = \frac{V_{oc}}{i_{sc}} \]
Finding the Thévenin Resistance Directly

We can find the Thévenin resistance by zeroing the sources in the original network and then computing the resistance between the terminals.

When zeroing a voltage source, it becomes a short circuit. When zeroing a current source, it becomes an open circuit.

Figure 2.45 When the source is zeroed, the resistance seen from the circuit terminals is equal to the Thévenin resistance.
Norton Equivalent Circuits

\[ I_n = i_{sc} \]

**Figure 2.49** The Norton equivalent circuit consists of an independent current source \( I_n \) in parallel with the Thévenin resistance \( R_t \).

**Figure 2.50** The Norton equivalent circuit with a short circuit across its terminals.
Step-by-step
Thévenin/Norton-Equivalent-Circuit Analysis

1. Perform two of these:
   a. Determine the open-circuit voltage \( V_t = v_{oc} \).
   b. Determine the short-circuit current \( I_n = i_{sc} \).
   c. Zero the sources and find the Thévenin resistance \( R_t \) looking back into the terminals.

2. Use the equation \( V_t = R_t \, I_n \) to compute the remaining value.

3. The Thévenin equivalent consists of a voltage source \( V_t \) in series with \( R_t \).

4. The Norton equivalent consists of a current source \( I_n \) in parallel with \( R_t \).
Source Transformations

Figure 2.53 A voltage source in series with a resistance is externally equivalent to a current source in parallel with the resistance, provided that $I_n = V_i / R_i$.  

Maximum Power Transfer

The load resistance that absorbs the maximum power from a two-terminal circuit is equal to the Thévenin resistance.
Superposition Principle

The **superposition principle** states that the total response is the sum of the responses to each of the independent sources acting individually. In equation form, this is

\[ r_T = r_1 + r_2 + \cdots + r_n \]

---

Figure 2.60 Circuit for Example 2.20 and Exercise 2.27.
Amplifiers

Basic Amplifier Concepts

![Diagram of an amplifier with input and output terminals, signal source, and load resistor. The equation $v_o(t) = Av_i(t)$ is shown, where $A$ is the gain.]

*Figure 11.1* Electronic amplifier.
Basic Amplifier Concepts

Ideally, an amplifier produces an output signal with identical waveshape as the input signal, but with a larger amplitude.

$$v_o(t) = A_v v_i(t)$$

Figure 11.2 Input waveform and corresponding output waveforms.
Inverting versus Non-inverting Amplifiers

Inverting amplifiers have negative voltage gain, and the output waveform is an inverted version of the input waveform. Non-inverting amplifiers have positive voltage gain.

Voltage-Amplifier Model

The input resistance $R_i$ is the equivalent resistance see when looking into the input terminals of the amplifier. $R_o$ is the output resistance. It causes the output voltage to decrease as the load resistance becomes smaller. $A_{voc}$ is the open circuit voltage gain.
Voltage-Amplifier Model

Thevenin equivalents

Current Gain

\[ A_i = \frac{i_o}{i_i} \quad A_i = \frac{i_o}{i_i} = \frac{v_o}{v_i} = A_v \frac{R_i}{R_L} \]
Power Gain

\[ G = \frac{P_o}{P_i} = \frac{V_o I_o}{V_i I_i} = A_v A_i = (A_v)^2 \frac{R_i}{R_L} \]

*Upper case V and I indicate Root Mean Square (RMS) values*

---

**Cascaded Amplifiers**

*Figure 11.5 Cascade connection of two amplifiers.*
The overall voltage gain of cascaded amplifiers is the product of the gain of the individual stages. This is also true for the current and power gains.

\[ A_v = \frac{v_{o2}}{v_{i1}} = \frac{v_{o1}}{v_{i1}} \cdot \frac{v_{o2}}{v_{i2}} = A_{v1} A_{v2} \]

The overall voltage gain of cascaded amplifiers is the product of the gain of the individual stages. This is also true for the current and power gains.

Simplified Models for Cascaded Amplifier Stages

First, determine the voltage gain of the first stage accounting for loading by the second stage.

The overall voltage gain is the product of the gains of the separate stages.

The input impedance is that of the first stage, and the output impedance is that of the last stage.
Op Amps

Ideal Operational Amplifiers

$\text{Figure 14.1} \quad \text{Circuit symbol for the op amp.}$
Characteristics of Ideal Op Amps

- Infinite gain for the differential input signal
- Zero gain for the common-mode input signal
- Infinite input impedances
- Zero output impedance
- Infinite bandwidth

Figure 14.2 Equivalent circuit for the ideal op amp. The open-loop gain $A_{OL}$ is very large (approaching infinity).
Operational amplifiers are almost always used with negative feedback, in which part of the output signal is returned to the input in opposition to the source signal.

In a negative feedback system, the ideal op-amp output voltage attains the value needed to force the differential input voltage and input current to zero. We call this fact the **summing-point constraint**.
Ideal op-amp circuits are analyzed by the following steps:

1. Verify that *negative* feedback is present.

2. Assume that the differential input voltage and the input current of the op amp are forced to zero. (This is the summing-point constraint.)

3. Apply standard circuit-analysis principles, such as Kirchhoff’s laws and Ohm’s law, to solve for the quantities of interest.
The Basic Inverter

![Figure 14.4 The inverting amplifier.]

Applying the Summing Point Constraint

![Figure 14.5 We make use of the summing-point constraint in the analysis of the inverting amplifier.]

\[ i_1 = \frac{v_{in}}{R_1} \]

\[ i_2 = \frac{v_o}{R_2} \]
Inverting Amplifier

![Inverting Amplifier Circuit Diagram]

\[ A_v = \frac{v_o}{v_{in}} = -\frac{R_2}{R_1} \]

Op-Amp Circuits
Summing Amplifier

\[ i_A = \frac{V_A}{R_A} \quad i_B = \frac{V_B}{R_B} \quad i_F = i_A + i_B = \frac{V_A}{R_A} + \frac{V_B}{R_B} \]

\[ i_F = \frac{0 - v_o}{R_F} = -\frac{v_o}{R_F} = \frac{V_A}{R_A} + \frac{V_B}{R_B} \quad \rightarrow \quad v_o = -R_F \left( \frac{V_A}{R_A} + \frac{V_B}{R_B} \right) \]
Summing Amplifier

Input resistance seen by $v_A = R_A$

Input resistance seen by $v_B = R_B$

Since the output voltage does not depend on the load resistance $R_L$, the output impedance is zero.

Non-Inverting Amplifier

$\begin{align*}
    v_i = 0 \quad \rightarrow \quad v_1 = v_{in} \\
    v_1 = \frac{R_1}{R_1 + R_2} v_o \quad \rightarrow \quad v_o = \frac{R_1 + R_2}{R_1} v_{in} = \left(1 + \frac{R_2}{R_1}\right) v_{in} \\
    A_v = \frac{v_o}{v_{in}} = \frac{1 + R_2}{R_1}
\end{align*}$
Non-Inverting Amplifier

Under the ideal-op-amp assumption, the non-inverting amplifier is an ideal voltage amplifier having infinite input resistance and zero output resistance.

\[ A_v = \frac{v_o}{v_{in}} = 1 + \frac{R_2}{R_1} \]

Voltage Follower

\[ A_v = \frac{v_o}{v_{in}} = 1 + \frac{R_2}{R_1} = 1 + \frac{0}{\infty} = 1 \]
Voltage-to-Current Converter

\[ i_o = \frac{v_{in}}{R_F} \]

Integrators and Differentiators

Integrators produce output voltages that are proportional to the running time integral of the input voltages. In a running time integral, the upper limit of integration is \( t \).
\[
\begin{align*}
&i_{in} = \frac{v_{in}}{R} \\
&v_c = \frac{q}{C} = \frac{1}{C} \int_0^t i_{in} \, dt = \frac{1}{RC} \int_0^t v_{in} \, dt \\
-&v_o - v_c = 0 \Rightarrow v_o = -v_c = -\frac{1}{RC} \int_0^t v_{in} \, dt
\end{align*}
\]

\[
v_o(t) = -\frac{1}{RC} \int_0^t v_{in}(t) \, dt
\]
Differentiator Circuit

\[ i_{in} = \frac{dq}{dt} = C \frac{dv_{in}}{dt} = \frac{0 - v_o}{R} \quad \Rightarrow v_o = -RC \frac{dv_{in}}{dt} \]

Differentiator Circuit

\[ v_o(t) = -RC \frac{dv_{in}}{dt} \]
Inductance and Capacitance

Energy is stored in the electric field that exists between the plates when the capacitor is charged.

$C = \frac{q}{v}$
Capacitance of the Parallel-Plate Capacitor

\[ C = \frac{\epsilon A}{d} \quad A = WL \]

\[ \epsilon_0 \approx 8.85 \times 10^{-12} \text{ F/m} \]

\[ \epsilon = \epsilon_r \epsilon_0 \]

Capacitance

\[ C = \frac{q}{v} \quad v = \frac{q}{C} \]

\[ q(t) = \int_{t_0}^{t} i(t) dt + q(t_0) \]

\[ v(t) = \frac{1}{C} \int_{t_0}^{t} i(t) dt + v(t_0) \]
Stored Energy

\[ p(t) = v(t)i(t) = vC \frac{dv}{dt} \]
\[ w(t) = \int_{t_0}^{t} p(t) \, dt = C \int_{t_0}^{t} v \frac{dv}{dt} \, dt = C \int_{0}^{v(t)} v \, dv = \frac{1}{2} Cv^2(t) = \frac{1}{2} v(t)q(t) = \frac{q^2(t)}{2C} \]
\[ C = \frac{q(t)}{v(t)} \quad v(t) = \frac{q(t)}{C} \]

Capacitances in Parallel

\[ i_1 = C_1 \frac{dV}{dt} \]
\[ i_2 = C_2 \frac{dV}{dt} \]
\[ i_3 = C_3 \frac{dV}{dt} \]
\[ i = i_1 + i_2 + i_3 = C_1 \frac{dV}{dt} + C_2 \frac{dV}{dt} + C_3 \frac{dV}{dt} = (C_1 + C_2 + C_3) \frac{dV}{dt} = C_{eq} \frac{dV}{dt} \]
\[ C_{eq} = C_1 + C_2 + C_3 \]
Capacitances in Series

\[ v = v_1 + v_2 + v_3 = \frac{1}{C_1} \int_0^t i(t) \, dt + \frac{1}{C_2} \int_0^t i(t) \, dt + \frac{1}{C_3} \int_0^t i(t) \, dt \]

\[ v = \frac{Q}{C_{eq}} = \frac{1}{C_{eq}} \int_0^t i(t) \, dt = \frac{1}{C_1} \int_0^t i(t) \, dt + \frac{1}{C_2} \int_0^t i(t) \, dt + \frac{1}{C_3} \int_0^t i(t) \, dt = \left[ \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \right] \int_0^t i(t) \, dt \]

\[ \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \]

Inductance

(a) Toroidal inductor
(b) Coil with an iron-oxide slug that can be screwed in or out to adjust the inductance
(c) Inductor with a laminated iron core
Inductance

\[ v(t) = L \frac{di}{dt} \]

\[ i(t) = \frac{1}{L} \int_{t_0}^{t} v(t)dt + i(t_0) \]

\[ p(t) = v(t)i(t) = Li(t) \frac{di}{dt} \]

\[ w(t) = \int_{0}^{t} p(t)dt = \int_{0}^{t} Li \frac{di}{dt}dt = \int_{0}^{i(t)} Lidi = \frac{1}{2} Li^2(t) \]

Series Inductances

\[ v(t) = L_{eq} \frac{di}{dt} = v_1 + v_2 + v_2 = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + L_3 \frac{di}{dt} = (L_1 + L_2 + L_3) \frac{di}{dt} \]

\[ L_{eq} = L_1 + L_2 + L_3 \]
Parallel Inductances

\[ i(t) = \frac{1}{L_{eq}} \int_0^t v(t) \, dt + i(t = 0) = i_1 + i_2 + i_3 \]

\[ = \frac{1}{L_1} \int_0^t v(t) \, dt + \frac{1}{L_2} \int_0^t v(t) \, dt + \frac{1}{L_3} \int_0^t v(t) \, dt = \left( \frac{1}{L_1 + L_2 + L_3} \right) \int_0^t v(t) \, dt \]

\[ \frac{1}{L_{eq}} = \frac{1}{L_1 + L_2 + L_3} \]

Mutual Inductance

Fields are aiding

\[ v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \]

\[ v_2 = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} \]

Fields are opposing

\[ v_1 = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} \]

\[ v_2 = -M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} \]

Magnetic flux produced by one coil links the other coil
First Order Transient Response

Transients

The time-varying currents and voltages resulting from the sudden application of sources, usually due to switching, are called transients. By writing circuit equations, we obtain integro-differential equations.
Discharge of a Capacitance through a Resistance

\[ RC \frac{dv_C(t)}{dt} + v_C(t) = 0 \quad \frac{dv_C(t)}{dt} = -\frac{1}{RC} v_C(t) \]

We need a function \( v_C(t) \) that has the same form as its derivative.

\[ v_C(t) = Ke^{st} \]

Substituting this in for \( v_c(t) \)

\[ RCKse^{st} + Ke^{st} = 0 \]

Discharge of a Capacitance through a Resistance

Solving for \( s \):

\[ s = -\frac{1}{RC} \]

Substituting into \( v_c(t) \):

\[ v_c(t) = Ke^{-t/RC} \]

\[ v_c(0+) = V_i \quad v_c(t) = V_ie^{-t/RC} \]
Discharge of a Capacitance through a Resistance

Charging a Capacitance from a DC Source through a Resistance

\[
C \frac{dv_C(t)}{dt} + \frac{v_C(t) - V_S}{R} = 0
\]

Rearranging:

\[
RC \frac{dv_C(t)}{dt} + v_C(t) = V_S
\]

This is a linear first-order differential equation with constant coefficients.
### Charging a Capacitance from a DC Source through a Resistance

The boundary conditions are given by the fact that the voltage across the capacitance cannot change instantaneously:

\[ v_C(0+) = v_C(0-) = 0 \]

### Try the solution

Try the solution:

\[ v_C(t) = K_1 + K_2 e^{st} \]

Substituting into the differential equation:

\[ RC \frac{dv_C(t)}{dt} + v_C(t) = V_S \]

Gives:

\[ (1 + RCs)K_2 e^{st} + K_1 = V_S \]
Charging a Capacitance from a DC Source through a Resistance

\[(1 + RC_s)K_2e^{st} + K_1 = V_S\]

For equality, the coefficient of \(e^{st}\) must be zero:

\[1 + RC_s = 0 \rightarrow s = \frac{-1}{RC}\]

Which gives \(K_1 = V_S\)

---

Charging a Capacitance from a DC Source through a Resistance

Substituting in for \(K_1\) and \(s\):

\[v_C(t) = K_1 + K_2e^{st} = V_S + K_2e^{-t/RC}\]

Evaluating at \(t=0\) and remembering that \(v_C(0+) = 0\)

\[v_C(0+) = V_S + K_2e^{0} = V_S + K_2 \rightarrow K_2 = -V_S\]

Substituting in for \(K_2\) gives:

\[v_C(t) = K_1 + K_2e^{st} = V_S - V_S e^{-t/RC}\]
Charging a Capacitance from a DC Source through a Resistance

In steady state, the voltage is constant, so the current through the capacitor is zero, so it behaves as an open circuit.

\[ v_C(t) = V_s - V_s e^{-t/\tau} \]

**Figure 4.4** The charging transient for the \( RC \) circuit of Figure 4.3.

DC Steady State

\[ i_C(t) = C \frac{dv_C(t)}{dt} \]

In steady state, the voltage is constant, so the current through the capacitor is zero, so it behaves as an open circuit.
DC Steady State

\[ v_L(t) = L \frac{di_L(t)}{dt} \]

In steady state, the current is constant, so the voltage across and inductor is zero, so it behaves as a short circuit.

---

DC Steady State

The steps in determining the forced response for \textit{RLC} circuits with dc sources are:

1. Replace capacitances with open circuits.
2. Replace inductances with short circuits.
3. Solve the remaining circuit.
RC/RL Circuits, Time Dependent Op Amp Circuits

RL Circuits

The steps involved in solving simple circuits containing dc sources, resistances, and one energy-storage element (inductance or capacitance) are:
1. Apply Kirchhoff’s current and voltage laws to write the circuit equation.

2. If the equation contains integrals, differentiate each term in the equation to produce a pure differential equation.

3. Assume a solution of the form $K_1 + K_2 e^{st}$.

4. Substitute the solution into the differential equation to determine the values of $K_1$ and $s$.

5. Use the initial conditions to determine the value of $K_2$.

6. Write the final solution.
Find $i(t)$ and the voltage $v(t)$

$i(t)= 0$ for $t < 0$ since the switch is open prior to $t = 0$

Apply KVL around the loop:

$$-V_S + i(t)R + v(t) = 0$$
RL Transient Analysis

Try \( i(t) = K_1 + K_2 e^{st} \)

\[
RK_1 + (RK_2 + sLK_2)e^{st} = V_s
\]

\[
RK_1 = V_s \Rightarrow K_1 = \frac{V_s}{R} = \frac{100V}{50\Omega} = 2A \quad RK_2 + sLK_2 = 0 \Rightarrow s = -\frac{L}{R}
\]

RL Transient Analysis

\[
i(t) = 2 + K_2 e^{-tR/L}
\]

\[
i(0^+) = 0 = 2 + K_2 e^0 = 2 + K_2 \Rightarrow K_2 = -2
\]

\[
i(t) = 2 - 2e^{-tR/L}
\]
RL Transient Analysis

Define \( \tau = \frac{L}{R} \)

\[ i(t) = 2 - 2e^{-t/\tau} \]

RC and RL Circuits with General Sources

First order differential equation with constant coefficients

\[ L \frac{di(t)}{dt} + Ri(t) = v_f(t) \]

\[ \frac{L}{R} \frac{di(t)}{dt} + i(t) = \frac{v_f(t)}{R} \]

\[ \tau \frac{dx(t)}{dt} + x(t) = f(t) \] Forcing function
RC and RL Circuits with General Sources

The general solution consists of two parts.

The particular solution (also called the forced response) is any expression that satisfies the equation.

\[ \tau \frac{dx(t)}{dt} + x(t) = f(t) \]

In order to have a solution that satisfies the initial conditions, we must add the complementary solution to the particular solution.
The homogeneous equation is obtained by setting the forcing function to zero.

$$\tau \frac{dx(t)}{dt} + x(t) = 0$$

The complementary solution (also called the natural response) is obtained by solving the homogeneous equation.

### Step-by-Step Solution

Circuits containing a resistance, a source, and an inductance (or a capacitance)

1. Write the circuit equation and reduce it to a first-order differential equation.
2. Find a particular solution. The details of this step depend on the form of the forcing function.

3. Obtain the complete solution by adding the particular solution to the complementary solution $x_c = Ke^{-t/\tau}$ which contains the arbitrary constant $K$.

4. Use initial conditions to find the value of $K$.

Second Order Transient Response
Damped Harmonic Motion

Mechanical oscillator

\[ m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = f_m(t) \]

Electrical oscillator

\[ L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = v_s(t) \]

Second –Order Circuits

\[ L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int_0^t i(t)dt + v_c(0) = v_s(t) \]

\[ \alpha = \frac{R}{2L} \]

Dampening coefficient

\[ \omega_0 = \frac{1}{\sqrt{LC}} \]

Undamped resonant frequency
Second –Order Circuits

\[
\frac{di(t)}{dt} + 2\alpha i(t) + \omega_0^2 \int_0^t i(t) dt + v_c(0) = v_s(t)
\]

Differentiate with respect to time:

\[
\frac{d^2i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = f(t)
\]

Solution of the Second-Order Equation

Particular solution

\[
\frac{d^2i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = f(t)
\]

Complementary solution

\[
\frac{d^2i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = 0
\]
Solution of the Complementary Equation

Try \( x_c(t) = Ke^{st} \):

\[
s^2 Ke^{st} + 2\alpha s Ke^{st} + \omega_0^2 Ke^{st} = 0
\]

Factoring:

\[
(s^2 + 2\alpha s + \omega_0^2)Ke^{st} = 0
\]

Characteristic equation:

\[
s^2 + 2\alpha s + \omega_0^2 = 0
\]

Solution of the Complementary Equation

\[
\zeta = \frac{\alpha}{\omega_0}
\]

Dampening ratio

Roots of the characteristic equation:

\[
s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}
\]
\[
s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}
\]
1. Overdamped case ($\zeta > 1$). If $\zeta > 1$ (or equivalently, if $\alpha > \omega_0$), the roots of the characteristic equation are real and distinct. Then the complementary solution is:

$$x_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

In this case, we say that the circuit is **overdamped**.

2. Critically damped case ($\zeta = 1$). If $\zeta = 1$ (or equivalently, if $\alpha = \omega_0$), the roots are real and equal. Then the complementary solution is

$$x_c(t) = K_1 e^{s_1 t} + K_2 te^{s_1 t}$$

In this case, we say that the circuit is **critically damped**.
3. Underdamped case ($\zeta < 1$). Finally, if $\zeta < 1$ (or equivalently, if $\alpha < \omega_0$), the roots are complex. (By the term complex, we mean that the roots involve the square root of $-1$.) In other words, the roots are of the form:

$$s_1 = -\alpha + j\omega_n \text{ and } s_2 = -\alpha - j\omega_n$$

in which $j$ is the square root of $-1$ and the natural frequency is given by:

$$\omega_n = \sqrt{\omega_0^2 - \alpha^2}$$

For complex roots, the complementary solution is of the form:

$$x_c(t) = K_1 e^{-\alpha t} \cos(\omega_n t) + K_2 e^{-\alpha t} \sin(\omega_n t)$$

In this case, we say that the circuit is underdamped.
We can replace the circuit with its Norton equivalent and then analyze the circuit by writing KCL at the top node:

\[
C \frac{dv(t)}{dt} + \frac{1}{R} v(t) + \frac{1}{L} \int_0^t v(t) dt + i_L(0) = i_n(t)
\]

Differentiating:

\[
C \frac{d^2v(t)}{dt^2} + \frac{1}{R} \frac{dv(t)}{dt} + \frac{1}{L} \frac{v(t)}{dt} = \frac{di_n(t)}{dt}
\]

\[
\frac{d^2v(t)}{dt^2} + \frac{1}{RC} \frac{dv(t)}{dt} + \frac{1}{LC} v(t) = \frac{1}{C} \frac{di_n(t)}{dt}
\]
Circuits with Parallel L and C

\[ \frac{d^2 v(t)}{dt^2} + \frac{1}{RC} \frac{dv(t)}{dt} + \frac{1}{LC} v(t) = \frac{1}{C} \frac{di_n(t)}{dt} \]

Dampening coefficient \( \alpha = \frac{1}{2RC} \)

Undamped resonant frequency \( \omega_0 = \frac{1}{\sqrt{LC}} \)

Forcing function \( f(t) = \frac{1}{C} \frac{di_n(t)}{dt} \)

\[ \frac{d^2 v(t)}{dt^2} + 2\alpha \frac{dv(t)}{dt} + \omega_0^2 v(t) = f(t) \]

This is the same equation as we found for the series LC circuit with the following changes for \( \alpha \):

Parallel circuit \( \alpha = \frac{1}{2RC} \)

Series circuit \( \alpha = \frac{R}{2L} \)
Circuits with Parallel L and C

To find the particular solution \( v_p(t) \) (steady state response) replace C with an open circuit and L with a short circuit.

Sinusoidal Signals, Complex Numbers, Phasors
Sinusoidal Currents and Voltages

\[ v(t) = V_m \cos(\omega t + \theta) \]

- \( V_m \) is the **peak value**
- \( \omega \) is the **angular frequency** in radians per second
- \( \theta \) is the **phase angle**
- \( T \) is the **period**

Hambley mixes units; \( \omega \) in radians, \( \theta \) in degrees

**Frequency**

\[ f = \frac{1}{T} \quad [Hz = \frac{cycles}{sec}] \]

**Angular frequency**

\[ \omega = \frac{2\pi}{T} \quad \left[\frac{radians}{sec}\right] \]

\[ \omega = 2\pi f \]

\[ \sin(z) = \cos(z - 90^\circ) \]
Root Mean Square (RMS) Values

\[ V_{\text{rms}} = \sqrt{\frac{1}{T} \int_0^T v^2(t) \, dt} \quad I_{\text{rms}} = \sqrt{\frac{1}{T} \int_0^T i^2(t) \, dt} \]

\[ P_{\text{avg}} = \frac{V_{\text{rms}}^2}{R} \quad P_{\text{avg}} = I_{\text{rms}}^2 R \]

RMS Value of a Sinusoid

\[ V_{\text{rms}} = \frac{V_m}{\sqrt{2}} \]

The rms value for a sinusoid is the peak value divided by the square root of two. This is not true for other periodic waveforms such as square waves or triangular waves.
**Phasor Definition**

Time function: \( v_1(t) = V_1 \cos(\omega t + \theta_1) \)

Phasor: \( V_1 = V_1 \angle \theta_1 \)

**Phasor Arithmetic**

\[
Z_1 = 10 \angle 60^\circ = 10 \cos(60^\circ) + j10 \sin(60^\circ) = 10 \cdot \frac{1}{2} + j10 \cdot \frac{\sqrt{3}}{2} = 5 + j8.66
\]

\[
Z_2 = 5 \angle 45^\circ = 5 \cos(45^\circ) + j5 \sin(45^\circ) = 5 \cdot \frac{\sqrt{2}}{2} + j5 \cdot \frac{\sqrt{2}}{2} = 3.54 + j3.54
\]

\[
Z_1 \times Z_2 = 10 \angle 60^\circ \times 5 \angle 45^\circ = 50 \angle 105^\circ
\]

\[
\frac{Z_1}{Z_2} = \frac{10 \angle 60^\circ}{5 \angle 45^\circ} = 2 \angle 15^\circ
\]

\[
Z_1 + Z_2 = 5 + j8.66 + 3.54 + j3.54 = 8.54 + j12.2
\]

\[
Z_1 - Z_2 = 5 + j8.66 - 3.54 - j3.54 = 1.46 + j5.12
\]
Adding Sinusoids Using Phasors

Step 1: Determine the phasor for each term.

Step 2: Add the phasors using complex arithmetic.

Step 3: Convert the sum to polar form.

Step 4: Write the result as a time function.

Phase Relationships

To determine phase relationships from a phasor diagram, consider the phasors to rotate counterclockwise. Then when standing at a fixed point, if $V_1$ arrives first followed by $V_2$ after a rotation of $\theta$, we say that $V_1$ leads $V_2$ by $\theta$. Alternatively, we could say that $V_2$ lags $V_1$ by $\theta$. (Usually, we take $\theta$ as the smaller angle between the two phasors.)
Phase Relationships

To determine phase relationships between sinusoids from their plots versus time, find the shortest time interval \( t_p \) between positive peaks of the two waveforms. Then, the phase angle is \( \theta = \left( \frac{t_p}{T} \right) \times 360^\circ \). If the peak of \( v_1(t) \) occurs first, we say that \( v_1(t) \) leads \( v_2(t) \) or that \( v_2(t) \) lags \( v_1(t) \).

Complex Impedances-Inductor

\[
\begin{align*}
    i_L(t) &= i_m \sin(\omega t + \theta) \\
    v_L(t) &= L \frac{di_L(t)}{dt} = \omega L i_m \cos(\omega t + \theta) \\
    I_L &= i_m \angle \theta - 90 \\
    V_L &= \omega L i_m \angle \theta = (\omega L \angle 90) I_L = j \omega L I_L \\
    Z_L &= j \omega L = \omega L \angle 90 \\
    V_L &= Z_L I_L
\end{align*}
\]
Complex Impedances-Inductor

\[ V_L = j\omega L \times I_L \]

\[ Z_L = j\omega L = \omega L \angle 90^\circ \]

\[ V_L = Z_L I_L \]

(a) Phasor diagram

(b) Current and voltage versus time

*Figure 5.7* Current lags voltage by 90° in a pure inductance.
Complex Impedances-Capacitor

\[ V_C = Z_C I_C \]

\[ Z_C = -j \frac{1}{\omega C} = \frac{1}{j \omega C} = \frac{1}{\omega C} \angle -90^\circ \]

\[ V_R = R I_R \]

(a) Phasor diagram
(b) Current and voltage versus time

Figure 5.8 Current leads voltage by 90° in a pure capacitance.
Impedances - Resistor

\[ V_R = R I_R \]

Figure 5.9 For a pure resistance, current and voltage are in phase.
Fourier Analysis, Low Pass Filters, Decibels

Figure 6.1 The short segment of a music waveform shown in (a) is the sum of the sinusoidal components shown in (b).
Fourier Analysis

All real-world signals are sums of sinusoidal components having various frequencies, amplitudes, and phases.

\[ v_{sq}(t) = \frac{4A}{\pi} \sin(\omega_0 t) + \frac{4A}{3\pi} \sin(3\omega_0 t) + \frac{4A}{5\pi} \sin(5\omega_0 t) + \ldots \]
Filters

Filters process the sinusoid components of an input signal differently depending of the frequency of each component. Often, the goal of the filter is to retain the components in certain frequency ranges and to reject components in other ranges.
Transfer Functions

The transfer function $H(f)$ of the two-port filter is defined to be the ratio of the phasor output voltage to the phasor input voltage as a function of frequency:

$$H(f) = \frac{V_{\text{out}}}{V_{\text{in}}}$$

Transfer Functions

The magnitude of the transfer function shows how the amplitude of each frequency component is affected by the filter. Similarly, the phase of the transfer function shows how the phase of each frequency component is affected by the filter.
Transfer Functions

Determining the output of a filter for an input with multiple components:

1. Determine the frequency and phasor representation for each input component.

2. Determine the (complex) value of the transfer function for each component.
3. Obtain the phasor for each output component by multiplying the phasor for each input component by the corresponding transfer-function value.

4. Convert the phasors for the output components into time functions of various frequencies. Add these time functions to produce the output.
Linear circuits behave as if they:

1. Separate the input signal into components having various frequencies.

2. Alter the amplitude and phase of each component depending on its frequency.

3. Add the altered components to produce the output signal.

First-Order Low Pass Filter

\[
I = \frac{V_{in}}{R + \frac{1}{j2\pi fC}}
\]

\[
V_{out} = \frac{1}{j2\pi C} I = \left( \frac{1}{j2\pi C} \right) \left( \frac{V_{in}}{R + \frac{1}{j2\pi C}} \right)
\]

\[
H(f) = \frac{V_{out}}{V_{in}} = \frac{1}{1 + j2\pi fRC} = \frac{1}{1 + j(f/f_B)}
\]

\[
f_B = \frac{1}{2\pi RC}
\]

Half power frequency
First-Order Low Pass Filter

\[ H(f) = \frac{1}{1 + j(f/f_B)} = \frac{1}{\sqrt{1 + (f/f_B)^2}} \arctan \left( \frac{f}{f_B} \right) \]

\[ |H(f)| = \frac{1}{\sqrt{1 + (f/f_B)^2}} \quad \angle H(f) = -\arctan \left( \frac{f}{f_B} \right) \]

For low frequency signals the magnitude of the transfer function is unity and the phase is 0°. Low frequency signals are passed while high frequency signals are attenuated and phase shifted.
First-Order Low Pass Filter

\[ I = \frac{V_{in}}{R + (2\pi fL)j} \]

\[ V_{out} = IR = \left( \frac{V_{in}}{R + (2\pi fL)j} \right) R \]

\[ H(f) = \frac{V_{out}}{V_{in}} = \frac{R}{R + j2\pi fL} = \frac{1}{1 + j\left(\frac{2\pi f}{R}\right)} = \frac{1}{1 + j(f/f_B)} \]

\[ f_B = \frac{R}{2\pi L} \]

---

**Figure 6.13** Cascade connection of two two-port circuits.

\[ H(f) = \frac{V_{out_2}}{V_{in_1}} = \frac{V_{out_1}}{V_{in_1}} \cdot \frac{V_{out_2}}{V_{out_1}} = \frac{V_{out_1}}{V_{in_1}} \cdot \frac{V_{out_2}}{V_{in_2}} = H_1(f)H_2(f) \]
Cascaded Two-Port Networks

\[ H(f) = H_1(f) \times H_2(f) \]

\[ |H(f)|_{dB} = |H_1(f)|_{dB} + |H_2(f)|_{dB} \]

Bode Plot, High Pass Filter
First-Order Low Pass Filter

\[
H(f) = \frac{1}{1 + j(f/f_B)} = \frac{1\angle0^\circ}{\sqrt{1+(f/f_B)^2} \arctan\left(\frac{f}{f_B}\right)}
\]

\[
|H(f)| = \frac{1}{\sqrt{1+(f/f_B)^2}} \quad \angle H(f) = -\arctan\left(\frac{f}{f_B}\right)
\]

Bode Plot for a First-Order Low-Pass Filter

A Bode plot shows the magnitude of a network function in decibels versus frequency using a logarithmic scale for frequency.

\[
|H(f)| = \frac{1}{\sqrt{1+(f/f_B)^2}}
\]

\[
|H(f)|_{\text{dB}} = -10 \log\left[1 + \left(\frac{f}{f_B}\right)^2\right]
\]
Bode Plot for a First-Order Low-Pass Filter

\[ |H(f)| = \frac{1}{\sqrt{1 + (f/f_B)^2}} \]

\[ |H(f)|_{db} = 20 \log |H(f)| = 20 \log \left( \frac{1}{\sqrt{1 + (f/f_B)^2}} \right) \]

\[ = 20 \log(1) - 20 \log \left( \sqrt{1 + (f/f_B)^2} \right) \]

\[ = -20 \log \left( \sqrt{1 + (f/f_B)^2} \right) \]

\[ = -20 \log \left( 1 + (f/f_B)^2 \right)^{1/2} \]

\[ = -10 \log \left( 1 + (f/f_B)^2 \right) \]

Asymptotic Behavior of Magnitude for Low and High Frequencies

\[ |H(f)|_{db} = -10 \log \left[ 1 + \left( \frac{f}{f_B} \right)^2 \right] \]

For \( f << f_B \) \hspace{1cm} |H(f)|_{db} \approx -10 \log(1) = 0

For \( f >> f_B \) \hspace{1cm} |H(f)|_{db} \approx -10 \log \left( \frac{f}{f_B} \right)^2 = -20 \log \left( \frac{f}{f_B} \right) \]
Magnitude Bode Plot for First-Order Low Pass Filter

![Magnitude Bode Plot](image)

Figure 6.15 Magnitude Bode plot for the first-order lowpass filter.

Asymptotic Behavior of Phase for Low and High Frequencies

\[
H(f) = \frac{1}{1 + j(f/f_B)} = \frac{1\angle 0^\circ}{\sqrt{1 + (f/f_B)^2}\angle \tan^{-1}(f/f_B)}
\]

\[
\angle H(f) = \tan^{-1}(f/f_B)
\]

- \(0 \quad f << f_B\)
- \(90 \quad f >> f_B\)
- \(45 \quad f = f_B\)
1. A horizontal line at zero for \( f < f_B / 10 \).
2. A sloping line from zero phase at \( f_B / 10 \) to \(-90^\circ\) at \( 10f_B \).
3. A horizontal line at \(-90^\circ\) for \( f > 10f_B \).
First-Order High-Pass Filter

\[ v_{\text{out}} = \frac{R}{R - j \frac{1}{2\pi fC}} v_{\text{in}} = \frac{1}{1 - j \frac{1}{2\pi fRC}} v_{\text{in}} = \frac{j(2\pi fRC)}{j2\pi fRC + 1} v_{\text{in}} \]

\[ \frac{v_{\text{out}}}{v_{\text{in}}} = \frac{j(f/f_B)}{1 + j(f/f_B)} \quad \text{where } f_B = \frac{1}{2\pi fRC} \]

First-Order High-Pass Filter

\[ H(f) = \frac{v_{\text{out}}}{v_{\text{in}}} = \frac{j(f/f_B)}{1 + j(f/f_B)} = \frac{(f/f_B)\angle 90^\circ}{\sqrt{1 + (f/f_B)^2} \angle \tan^{-1}(f/f_B)} \]

\[ |H(f)| = \frac{(f/f_B)}{\sqrt{1 + (f/f_B)^2}} \]

\[ \angle H(f) = \frac{\angle 90^\circ}{\angle \tan^{-1}(f/f_B)} = 90^\circ - \tan^{-1}(f/f_B) \]
First-Order High-Pass Filter

\[ |H(f)| = \frac{f/f_B}{\sqrt{1 + (f/f_B)^2}} \]

\[ \angle H(f) = 90^\circ - \tan^{-1}\left(\frac{f}{f_B}\right) \]

First-Order High-Pass Filter

\[ |H(f)| = \frac{f/f_B}{\sqrt{1 + (f/f_B)^2}} \]

\[ |H(f)|_{dB} = 20 \log|H(f)| = 20 \log\left(\frac{f/f_B}{\sqrt{1 + (f/f_B)^2}}\right) \]

\[ = 20 \log(f/f_B) - 20 \log\left(\sqrt{1 + (f/f_B)^2}\right) \]

\[ = 20 \log(f/f_B) - 20 \log\left(1 + (f/f_B)^2\right)^{1/2} \]

\[ = 20 \log(f/f_B) - 10 \log\left(1 + (f/f_B)^2\right) \]
Asymptotic Behavior of First-Order High-Pass Filter

\[ |H(f)| = 20 \log\left(\frac{f}{f_B}\right) - 10 \log\left(1 + \left(\frac{f}{f_B}\right)^2\right) \]

For \( f << f_B \) \( |H(f)| \approx 20 \log\left(\frac{f}{f_B}\right) \)

For \( f >> f_B \) \( |H(f)| \approx 20 \log\left(\frac{f}{f_B}\right) - 10 \log\left(\frac{f}{f_B}\right)^2 = 0 \)

\[ \angle H(f) = 90^\circ - \arctan\left(\frac{f}{f_B}\right) \]

For \( f << f_B \) \( \angle H(f) \approx 90^\circ \)

For \( f >> f_B \) \( \angle H(f) \approx 0^\circ \)

Bode Plots for the First-Order High-Pass Filter

\[ |H(f)| = 20 \log\left(\frac{f}{f_B}\right) - 10 \log\left(1 + \left(\frac{f}{f_B}\right)^2\right) \]

For \( f << f_B \) \( |H(f)| \approx 20 \log\left(\frac{f}{f_B}\right) \)

For \( f >> f_B \) \( |H(f)| \approx 20 \log\left(\frac{f}{f_B}\right) - 10 \log\left(\frac{f}{f_B}\right)^2 = 0 \)

\[ \angle H(f) = 90^\circ - \arctan\left(\frac{f}{f_B}\right) \]

For \( f << f_B \) \( \angle H(f) \approx 90^\circ \)

For \( f >> f_B \) \( \angle H(f) \approx 0^\circ \)
First-Order High-Pass Filter

\[ V_{out} = \frac{j 2\pi f L}{R + j 2\pi L} \quad V_{in} = \frac{j 2\pi f L / R}{1 + j 2\pi f L / R} \quad V_{in} = \frac{j (f / f_B)}{1 + j (f / f_B)} V_{in} \quad \text{where } f_B = R / 2\pi L. \]

\[ H(f) = \frac{V_{out}}{V_{in}} = \frac{j (f / f_B)}{1 + j (f / f_B)} \]

High Pass Filters, 2\textsuperscript{nd} Order Filters, Active Filters, Resonances
Series Resonance

\[ Z_s(f) = j2\pi fL + R - j\frac{1}{2\pi fC} \]

For resonance the reactance of the inductor and the capacitor cancel:

\[ 2\pi f_0 L = \frac{1}{2\pi f_0 C} \rightarrow f_0^2 = \frac{1}{(2\pi)^2 LC} \]

\[ f_0 = \frac{1}{2\pi \sqrt{LC}} \]

Series Resonance

**Quality factor** \( Q_s \)

\[ Q_s = \frac{\text{Reactance of inductance at resonance}}{\text{Resistance}} \]

\[ = \frac{2\pi f_0 L}{R} \]

Substitute \( L = \frac{1}{(2\pi)^2 (f_0)^2 C} \) from \( f_0 = \frac{1}{2\pi \sqrt{LC}} \)

\[ Q_s = \frac{1}{2\pi f_0 CR} \]
Series Resonance

\[ Z_s(f) = R + j2\pi fL - j\frac{1}{2\pi fC} \]

\[ = R \left[ 1 + j \left( \frac{2\pi fL}{R} - \frac{1}{2\pi fRC} \right) \right] \]

\[ = R \left[ 1 + j \frac{2\pi f_0 L}{R} \left( \frac{f}{f_0} - \frac{1}{(2\pi)^2 f f_0 LC} \right) \right] \]

Substitute \( f_0 = \frac{1}{2\pi \sqrt{LC}} \) and \( Q_s = \frac{2\pi f_0 L}{R} \)

\[ Z_s(f) = R \left[ 1 + j Q_s \left( \frac{f}{f_0} - \frac{f_0}{f} \right) \right] \]

---

**Figure 6.26** Plots of normalized magnitude and phase for the impedance of the series resonant circuit versus frequency.

\[
\left| \frac{Z}{R} \right| = \sqrt{1 + Q_s^2 \left( \frac{f}{f_0} - \frac{f_0}{f} \right)^2} \quad \angle Z_s = \tan^{-1} Q_s \left( \frac{f}{f_0} - \frac{f_0}{f} \right)
\]
Series Resonant Band-Pass Filter

\[ I = \frac{V_s}{Z_s(f)} = \frac{V_s / R}{1 + jQ_s \left( \frac{f - f_0}{f_0} \right)} \]

\[ V_R = RI = \frac{V_s}{1 + jQ_s \left( \frac{f - f_0}{f_0} \right)} \]

\[ V_R = \frac{1}{1 + jQ_s \left( \frac{f}{f_0} - \frac{f_0}{f} \right)} \]

Series Resonant Band-Pass Filter

\[ \frac{V_R}{V_s} = \frac{1}{1 + jQ_s \left( \frac{f}{f_0} - \frac{f_0}{f} \right)} \]
Series Resonant Band-Pass Filter

\[ B = f_H - f_L = \frac{f_0}{Q_s} \]

\[ f_L \approx f_0 - \frac{B}{2} \quad f_H \approx f_0 + \frac{B}{2} \]

Figure 6.28 The bandwidth \( B \) is equal to the difference between the half-power frequencies.
Parallel Resonance

\[ Z_p = \frac{1}{(1/R) + j2\pi fC - j(1/2\pi fL)} \]

At resonance \( Z_p \) is purely resistive:

\[ j2\pi f_0 C = j(1/2\pi f_0 L) \rightarrow f_0 = \frac{1}{2\pi \sqrt{LC}} \]

Parallel Resonance

**Quality factor** \( Q_p \)

\[ Q_p = \left( \frac{\text{Reactance of inductance at resonance}}{\text{Resistance}} \right)^{-1} \]

\[ = \frac{R}{2\pi f_0 L} \]

Substitute \( L = \frac{1}{(2\pi)^2 (f_0)^2 C} \) from \( f_0 = \frac{1}{2\pi \sqrt{LC}} \)

\[ Q_p = 2\pi f_0 CR \]
Parallel Resonance

\[ Z_p = \frac{1}{\frac{1}{R} + j2\pi f C - j\frac{1}{\sqrt{L}} + j2\pi f R C - j2\pi f L} = \frac{R}{1 + j2\pi f R C - j2\pi f L} \]

\[ = \frac{R}{1 + j2\pi f_0 R C \left( \frac{f}{f_0} - \frac{1}{(2\pi)^2 f_0 f L} \right)} = \frac{R}{1 + jQ_p \left( \frac{f}{f_0} - \frac{1}{(2\pi)^2 f_0 f L} \right)} \]

\[ = \frac{R}{1 + jQ_p \left( \frac{f}{f_0} - \frac{f_0}{f} \right)} \]

\[ V_{out} = IZ_p = \frac{IR}{1 + jQ_p \left( \frac{f}{f_0} - \frac{f_0}{f} \right)} \]

\[ V_{out} \text{ for constant current, varyin} \text{ the frequency} \]
**Ideal Filters**

![Graphs of ideal filters: lowpass, highpass, bandpass, and band reject](image)

**Second Order Low-Pass Filter**

\[
\begin{align*}
V_{out} &= \frac{Z_C}{Z_R + Z_L + Z_C} V_{in} = \frac{-j}{2\pi C} \frac{2\pi R}{R + j2\pi L - \frac{j}{2\pi C}} V_{in} = \frac{-j}{2\pi R C} \left( 1 + j \frac{2\pi f L}{R} \left( \frac{f}{f_0} - \frac{1}{2\pi f_0 L C} \right) \right) V_{in} \\
V_{out} &= H(f) = \frac{-j}{2\pi R C} \left( 1 + j \frac{2\pi f L}{R} \left( \frac{f}{f_0} - \frac{1}{2\pi f_0 L C} \right) \right) = -jQ_3 \left( \frac{f}{f_0} \right) \left( \frac{f}{f_0} - \frac{1}{2\pi f_0 L C} \right)
\end{align*}
\]
Second-Order Low-Pass Filter

\[
H(f) = \frac{V_o}{V_{in}} = \frac{-jQ_s(f_0/f)}{1 + jQ_s(f/f_0 - f_0/f)}
\]

\[
= \frac{Q_s(f_0/f)\angle -90^\circ}{\sqrt{1 + Q_s^2(f/f_0 - f_0/f)^2}} \angle Tan^{-1}Q_s(f/f_0 - f_0/f)
\]

\[
|H(f)| = \frac{Q_s(f_0/f)}{\sqrt{1 + Q_s^2(f/f_0 - f_0/f)^2}}
\]
Second Order High-Pass Filter

At low frequency the capacitor is an open circuit.

At high frequency the capacitor is a short and the inductor is open.

Second Order Band-Pass Filter

At low frequency the capacitor is an open circuit.

At high frequency the inductor is an open circuit.
Second Order Band-Reject Filter

At low frequency the capacitor is an open circuit.

At high frequency the inductor is an open circuit.

First-Order Low-Pass Filter

\[ H(f) = \frac{V_o}{V_i} = -\frac{Z_f}{Z_i} \]

\[ \frac{1}{Z_f} = \frac{1}{R_f} + \frac{1}{j2\pi C_f} = \frac{1}{R_f} + \frac{j2\pi R_f C_f}{R_f} \]

\[ Z_f = \frac{R_f}{1 + j2\pi R_f C_f} \]

\[ H(f) = \frac{Z_f}{Z_i} = \left( \frac{R_f}{R_i} \right) \frac{1}{1 + j2\pi R_f C_f} \]

\[ f_B = \frac{1}{2\pi R_f C_f} \]

A low-pass filter with a dc gain of \(-\frac{R_f}{R_i}\).
First-Order High-Pass Filter

\[ H(f) = \frac{v_o}{v_i} = -\frac{Z_f}{Z_i} \]

\[ Z_i = R_i + \frac{1}{j2\pi f C_i} \quad \quad Z_f = R_f \]

\[ H(f) = -\frac{Z_f}{Z_i} = -\frac{R_f}{R_i + \frac{1}{j2\pi f C_i}} \]

\[ = \frac{j2\pi f R_f C_i}{1 + j2\pi f R_i C_i} = \frac{R_f}{R_i} \frac{j2\pi f R_f C_i}{1 + j2\pi f R_i C_i} \]

\[ = \frac{R_f}{R_i} \left[ \frac{j(f/f_B)}{1 + j(f/f_B)} \right] \]

\[ f_B = \frac{1}{2\pi R_i C_i} \]

A high-pass filter with a high frequency gain of \(-R_f/R_i\)

Higher Order Filters

\[ H(f) = H_1(f)H_2(f)\cdots H_n(f) \]

\[ = (-1)^n \left( \frac{R_f}{R_i} \right)^n \left[ \frac{1}{1 + j(f/f_B)} \right]^n \]
Butterworth Transfer Function

Butterworth filters are characterized by having a particularly flat pass-band.

\[
|H(f)| = \frac{H_0}{\sqrt{1 + \left(\frac{f}{f_B}\right)^{2n}}}
\]

Magnetic Circuits, Materials
Magnetic Field Lines

Magnetic fields can be visualized as lines of flux that form closed paths.

The flux density vector $\mathbf{B}$ is tangent to the lines of flux.

$B = \text{Magnetic flux density}$

Right-Hand Rule

(a) If a wire is grasped with the thumb pointing in the current direction, the fingers encircle the wire in the direction of the magnetic field.

(b) If a coil is grasped with the fingers pointing in the current direction, the thumb points in the direction of the magnetic field inside the coil.
Flux Linkages and Faraday’s Law

Magnetic flux passing through a surface area A:

$$\phi = \int_A B \cdot dA$$

For a constant magnetic flux density perpendicular to the surface:

$$= BA$$

The flux linking a coil with N turns:

$$\lambda = N\phi$$

Faraday’s Law

Faraday’s law of magnetic induction:

$$e = \frac{d\lambda}{dt}$$

The voltage induced in a coil whenever its flux linkages are changing. Changes occur from:

- Magnetic field changing in time
- Coil moving relative to magnetic field
Lenz’s Law

**Lenz’s law** states that the polarity of the induced voltage is such that the voltage would produce a current (through an external resistance) that opposes the original change in flux linkages.

---

**Figure 15.4** When the flux linking a coil changes, a voltage is induced in the coil. The polarity of the voltage is such that if a circuit is formed by placing a resistance across the coil terminals, the resulting current produces a field that tends to oppose the original change in the field.
Magnetic Field Intensity and Ampère’s Law

\[ \mathbf{B} = \mu \mathbf{H} \quad H = \text{Magnetic field intensity} \]

\[ \mu_0 = 4\pi \times 10^{-7} \text{ W/Am} \]

\[ \mu_r = \frac{\mu}{\mu_0} \quad \text{Relative permeability} \]

Ampère’s Law: \[ \oint \mathbf{H} \cdot d\mathbf{l} = \sum i \]

Ampère’s Law

The line integral of the magnetic field intensity around a closed path is equal to the sum of the currents flowing through the surface bounded by the path.
Magnetic Field Around a Long Straight Wire

\[ Hl = H \cdot 2\pi r = I \]
\[ H = \frac{I}{2\pi r} \]
\[ B = \mu H = \frac{\mu I}{2\pi r} \]

Flux Density in a Toroidal Core

\[ Hl = H \cdot 2\pi R = NI \]
\[ H = \frac{NI}{2\pi R} \]
\[ B = \frac{\mu NI}{2\pi R} \]
Flux Density in a Toroidal Core

\[ \phi = BA = \frac{\mu NI}{2\pi R} \pi r^2 \]

\[ = \frac{\mu NI r^2}{2R} \]

\[ \lambda = N\phi = \frac{\mu N^2 I r^2}{2R} \]

Magnetic Circuits

In many engineering applications, we need to compute the magnetic fields for structures that lack sufficient symmetry for straight-forward application of Ampère’s law. Then, we use an approximate method known as magnetic-circuit analysis.
**magnetomotive force** (mmf) of an \(N\)-turn current-carrying coil

\[ \mathcal{F} = NI \quad \text{Analog: Voltage (emf)} \]

**reluctance** of a path for magnetic flux

\[ R = \frac{\ell}{\mu A} \quad \text{Analog: Resistance} \]

\[ \mathcal{F} = R\phi \quad \text{Analog: Ohm’s Law} \]

---

**Magnetic Circuit for Toroidal Coil**

\[ l = 2\pi R \quad A = \pi r^2 \]

\[ R = \frac{1}{\mu} \left( \frac{l}{A} \right) = \frac{1}{\mu} \left( \frac{2\pi R}{\pi r^2} \right) = \frac{1}{\mu} \left( \frac{2R}{r^2} \right) \]

\[ \mathcal{F} = NI \]

\[ \phi = \frac{\mathcal{F}}{R} = \frac{\mu Nr^2 I}{2R} \]
Advantage of the Magnetic-Circuit Approach

The advantage of the magnetic-circuit approach is that it can be applied to unsymmetrical magnetic cores with multiple coils.

A Magnetic Circuit with Reluctances in Series and Parallel

Find the flux density in each gap
A Magnetic Circuit with Reluctances in Series and Parallel

\[ R_{total} = R_c + \frac{1}{\frac{1}{R_a} + \frac{1}{R_b}} \]

\[ \phi_c = \frac{N_i}{R_{total}} \]

\[ \phi_a = \frac{R_b}{R_a + R_b} \phi_c \quad \text{(current divider)} \]

\[ \phi_b = \frac{R_a}{R_a + R_b} \phi_c \]

\[ B_a = \frac{\phi_a}{A_a} \]

\[ B_a = \frac{\phi_a}{A_a} \]

Mutual Inductance & Transformers
Inductance and Mutual Inductance

Definition of inductance $L$:

$$L = \frac{\text{Flux linkages}}{\text{current}} = \frac{\lambda}{i}$$

Substitute for the flux linkages using $\lambda = N\phi$

$$L = \frac{N\phi}{i}$$

Inductance and Mutual Inductance

Substituting $\phi = \frac{Ni}{R}$

$$L = \frac{N^2}{R}$$
Faraday’s Law

Voltage is induced in a coil when its flux linkages change:

\[ e = \frac{d\lambda}{dt} = \frac{d(Li)}{dt} = L \frac{di}{dt} \]

Mutual Inductance

Self inductance for coil 1

\[ L_1 = \frac{\lambda_{11}}{i_1} \]

Self inductance for coil 2

\[ L_2 = \frac{\lambda_{22}}{i_2} \]

Mutual inductance between coils 1 and 2:

\[ M = \frac{\lambda_{21}}{i_1} = \frac{\lambda_{12}}{i_2} \]
Mutual Inductance

Total fluxes linking the coils:

\[ \lambda_1 = \lambda_{11} \pm \lambda_{12} \]
\[ \lambda_2 = \lambda_{22} \pm \lambda_{21} \]

Currents entering the dotted terminals produce aiding fluxes.
Circuit Equations for Mutual Inductance

\[ \lambda_1 = L_1 i_1 \pm Mi_2 \]
\[ \lambda_2 = \pm Mi_1 + L_2 i_2 \]
\[ e_1 = \frac{d\lambda_1}{dt} = L_1 \frac{di_1}{dt} \pm M \frac{di_2}{dt} \]
\[ e_2 = \frac{d\lambda_2}{dt} = \pm M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} \]

Transformers

Can be used to *step up* or *step down* ac voltages
Ideal Transformers

\[ v_2(t) = \frac{N_2}{N_1} v_1(t) \]

\[ i_2(t) = \frac{N_1}{N_2} i_1(t) \]

\[ p_2(t) = p_1(t) \]

Mechanical Analog

\[ v_2(t) = \frac{N_2}{N_1} v_1(t) \]

\[ i_2(t) = \frac{N_1}{N_2} i_1(t) \]

\[ v_2 = \frac{l_2}{l_1} v_1 \]

\[ F_2 = \frac{l_1}{l_2} F_1 \]
Impedance Transformations

$$Z_L' = \frac{V_1}{I_1} = \left(\frac{N_1}{N_2}\right)^2 Z_L$$