EE 135, Winter 2012

Reading: start Chapter 3, sections 1-7.

Homework #3: problems, due today, 1/31/12
Chap.2, 2.28, 2.32(a-e), 2.38, 2.43, 2.77.

Discussion Session: 7-8pm Wednesdays, Jack’s Lounge

Prelim. 1 on Feb.2, 2012, Thursday. in class.
Covers chapters 1 and 2. only material assigned for reading.

Lecture 7
First Preliminary Exam

Next Thursday, Feb. 2, 2011
In class
Covers First three homework sets
Material from Chapters 1 and 2
(that was assigned as reading)

Closed book
One sheet of formulae (both sides)
(to be turned in with exam)
Show all your work and
Enclose answer in a box.
First Preliminary Exam

TOPICS:

- The Electromagnetic spectrum
- Traveling and Standing waves
- Phasors and Complex Algebra
- Waves in lossless and lossy media
- Lumped element model of transmission lines
- Wave propagation in transmission lines
- Lossless transmission line
- Voltage and Reflection coefficients
- Standing wave ratio
- Wave impedance
- Characteristic impedance
- Power flow in transmission lines
- Transients in transmission lines
3. VECTOR ANALYSIS

Applied EM by Ulaby, Michielssen and Ravaiolı
Chapter 3 Overview

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Objectives

Upon learning the material presented in this chapter, you should be able to:

1. Use vector algebra in Cartesian, cylindrical, and spherical coordinate systems.
2. Transform vectors between the three primary coordinate systems.
3. Calculate the gradient of a scalar function and the divergence and curl of a vector function in any of the three primary coordinate systems.
4. Apply the divergence theorem and Stokes’s theorem.
Laws of Vector Algebra

\[ A = \hat{a}|A| = \hat{a}A \]

\[ A = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z \]

\[ A = |A| = \sqrt{A_x^2 + A_y^2 + A_z^2} \]

\[ \hat{a} = \frac{A}{A} = \frac{\hat{x}A_x + \hat{y}A_y + \hat{z}A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \]
Properties of Vector Operations

Equality of Two Vectors

\[ \mathbf{A} = \hat{\mathbf{a}} \mathbf{A} = \hat{x} A_x + \hat{y} A_y + \hat{z} A_z, \quad (3.6a) \]
\[ \mathbf{B} = \hat{\mathbf{b}} \mathbf{B} = \hat{x} B_x + \hat{y} B_y + \hat{z} B_z, \quad (3.6b) \]

then \( \mathbf{A} = \mathbf{B} \) if and only if \( A = B \) and \( \hat{\mathbf{a}} = \hat{\mathbf{b}} \), which requires that \( A_x = B_x, \ A_y = B_y, \) and \( A_z = B_z \).

Equality of two vectors does not necessarily imply that they are identical; in Cartesian coordinates, two displaced parallel vectors of equal magnitude and pointing in the same direction are equal, but they are identical only if they lie on top of one another.

**Commutative property**

\[ \mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \]

**Figure 3-3:** Vector addition by (a) the parallelogram rule and (b) the head-to-tail rule.
Position & Distance Vectors

**Position Vector:** From origin to point P

\[
R_1 = \overrightarrow{OP_1} = \hat{x}x_1 + \hat{y}y_1 + \hat{z}z_1 \\
R_2 = \overrightarrow{OP_2} = \hat{x}x_2 + \hat{y}y_2 + \hat{z}z_2
\]

**Distance Vector:** Between two points

\[
R_{12} = \overrightarrow{P_1P_2} = R_2 - R_1 \\
= \hat{x}(x_2 - x_1) + \hat{y}(y_2 - y_1) + \hat{z}(z_2 - z_1)
\]

and the distance \(d\) between \(P_1\) and \(P_2\) equals the magnitude of \(R_{12}\):

\[
d = |R_{12}| \\
= \left[ (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \right]^{1/2}. \tag{3.12}
\]

*Figure 3-4:* Distance vector \(R_{12} = \overrightarrow{P_1P_2} = R_2 - R_1\), where \(R_1\) and \(R_2\) are the position vectors of points \(P_1\) and \(P_2\), respectively.
Vector Multiplication: Scalar Product or "Dot Product"

\[
A \cdot B = AB \cos \theta_{AB}
\]

Hence:

\[
A = |A| = \sqrt{A \cdot A}
\]

\[
\theta_{AB} = \cos^{-1} \left[ \frac{A \cdot B}{\sqrt{A \cdot A} \sqrt{B \cdot B}} \right]
\]

\[
\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1,
\]

\[
\hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{x} = 0.
\]

Figure 3-5: The angle \( \theta_{AB} \) is the angle between \( \mathbf{A} \) and \( \mathbf{B} \), measured from \( \mathbf{A} \) to \( \mathbf{B} \) between vector tails. The dot product is positive if \( 0 \leq \theta_{AB} < 90^\circ \), as in (a), and it is negative if \( 90^\circ < \theta_{AB} \leq 180^\circ \), as in (b).

\[
A \cdot B = B \cdot A \quad \text{(commutative property)},
\]

\[
A \cdot (B + C) = A \cdot B + A \cdot C \quad \text{(distributive property)}
\]

Hence:

\[
A \cdot B = A_x B_x + A_y B_y + A_z B_z.
\]
Vector Multiplication: Vector Product or "Cross Product"

\[ \mathbf{A} \times \mathbf{B} = \hat{n} AB \sin \theta_{AB} \]

\[ \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \quad \text{(anticommutative)} \]

\[ \mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \quad \text{(distributive)} \]

\[ \mathbf{A} \times \mathbf{A} = 0 \]

\[ \hat{x} \times \hat{y} = \hat{z}, \quad \hat{y} \times \hat{z} = \hat{x}, \quad \hat{z} \times \hat{x} = \hat{y}. \quad (3.25) \]

Note the cyclic order \((xyzxyz\ldots)\). Also,

\[ \hat{x} \times \hat{x} = \hat{y} \times \hat{y} = \hat{z} \times \hat{z} = 0. \quad (3.26) \]

If \( \mathbf{A} = (A_x, A_y, A_z) \) and \( \mathbf{B} = (B_x, B_y, B_z) \),

\[ \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \]
Example 3-1: Vectors and Angles

In Cartesian coordinates, vector \( \mathbf{A} \) points from the origin to point \( P_1 = (2, 3, 3) \), and vector \( \mathbf{B} \) is directed from \( P_1 \) to point \( P_2 = (1, -2, 2) \). Find

(a) vector \( \mathbf{A} \), its magnitude \( A \), and unit vector \( \hat{\mathbf{a}} \),
(b) the angle between \( \mathbf{A} \) and the \( y \)-axis,
(c) vector \( \mathbf{B} \),
(d) the angle \( \theta_{AB} \) between \( \mathbf{A} \) and \( \mathbf{B} \), and
(e) the perpendicular distance from the origin to vector \( \mathbf{B} \).

Solution: (a) Vector \( \mathbf{A} \) is given by the position vector of \( P_1 = (2, 3, 3) \) as shown in Fig. 3-7. Thus,

\[
\mathbf{A} = \hat{x}2 + \hat{y}3 + \hat{z}3,
\]

\[
A = |\mathbf{A}| = \sqrt{2^2 + 3^2 + 3^2} = \sqrt{22},
\]

\[
\hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = (\hat{x}2 + \hat{y}3 + \hat{z}3)/\sqrt{22}.
\]

(b) The angle \( \beta \) between \( \mathbf{A} \) and the \( y \)-axis is obtained from

\[
\mathbf{A} \cdot \hat{\mathbf{y}} = |A||\hat{\mathbf{y}}| \cos \beta = A \cos \beta,
\]

or

\[
\beta = \cos^{-1} \left( \frac{\mathbf{A} \cdot \hat{\mathbf{y}}}{A} \right) = \cos^{-1} \left( \frac{3}{\sqrt{22}} \right) = 50.2^\circ.
\]

(c)

\[
\mathbf{B} = \hat{x}(1 - 2) + \hat{y}(-2 - 3) + \hat{z}(2 - 3) = -\hat{x} - \hat{y}5 - \hat{z}.
\]

(d)

\[
\theta_{AB} = \cos^{-1} \left( \frac{\mathbf{A} \cdot \mathbf{B}}{|A||B|} \right) = \cos^{-1} \left( \frac{(-2 - 15 - 3)}{\sqrt{22} \sqrt{27}} \right)
\]

\[
= 145.1^\circ.
\]

(e) The perpendicular distance between the origin and vector \( \mathbf{B} \) is the distance \( |\overrightarrow{OP_3}| \) shown in Fig. 3-7. From right triangle \( \overrightarrow{OP_1P_3} \),

\[
|\overrightarrow{OP_3}| = |\mathbf{A}| \sin(180^\circ - \theta_{AB})
\]

\[
= \sqrt{22} \sin(180^\circ - 145.1^\circ) = 2.68.
\]

Figure 3-7: Geometry of Example 3-1.
Triple Products

Scalar Triple Product

\[ A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B). \]

Vector Triple Product

\[ A \times (B \times C) = B(A \cdot C) - C(A \cdot B), \]

which is known as the “bac-cab” rule.

Example 3-2: Vector Triple Product

Given \( A = \hat{x} - \hat{y} + 2\hat{z}, \) \( B = \hat{y} + 2\hat{z}, \) and \( C = -\hat{x} + 2\hat{z}, \) find \( (A \times B) \times C \) and compare it with \( A \times (B \times C). \)

Solution:

\[
A \times B = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
1 & -1 & 2 \\
0 & 1 & 1 \\
\end{vmatrix} = -\hat{x}3 - \hat{y} + \hat{z}
\]

and

\[
(A \times B) \times C = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
-3 & -1 & 1 \\
-2 & 0 & 3 \\
\end{vmatrix} = -\hat{x}3 + \hat{y}7 - 2\hat{z}.
\]

A similar procedure gives \( A \times (B \times C) = \hat{x}2 + \hat{y}4 + \hat{z}. \)

Hence:

\[ A \times (B \times C) \neq (A \times B) \times C \]
**Triple Product**

\[ \mathbf{A} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}, \quad \mathbf{B} = \mathbf{j} + \mathbf{k}, \quad \mathbf{C} = -2\mathbf{i} + 3\mathbf{k} \]

\[ \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i}(-1 - 2) + \mathbf{j}(2 - 0) + \mathbf{k}(1 - 0) = -3\mathbf{i} + \mathbf{j} + 2\mathbf{k} \]

\[ \mathbf{B} \times \mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ -2 & 0 & 3 \end{vmatrix} = \mathbf{i}(3 - 0) + \mathbf{j}(-2 + 0) + \mathbf{k}(0 + 2) = 3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \]

\[ \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 3 & -2 & 2 \end{vmatrix} = \mathbf{i}(2 + 4) - \mathbf{j}(2 - 0) + \mathbf{k}(-2 + 3) = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k} \]

\[ (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ -3 & 1 & 1 \end{vmatrix} = \mathbf{i}(3 - 0) - \mathbf{j}(-2 + 0) + \mathbf{k}(0 + 2) = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} \]

\[ \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \]
Cartesian Coordinate System

Differential length vector

\[ dl = \hat{x} \, dl_x + \hat{y} \, dl_y + \hat{z} \, dl_z = \hat{x} \, dx + \hat{y} \, dy + \hat{z} \, dz, \quad (3.34) \]

where \( dl_x = dx \) is a differential length along \( \hat{x} \), and similar interpretations apply to \( dl_y = dy \) and \( dl_z = dz \).

Differential area vectors

\[ ds_x = \hat{x} \, dl_y \, dl_z = \hat{x} \, dy \, dz \quad (y-z \text{ plane}), \quad (3.35a) \]

with the subscript on \( ds \) denoting its direction. Similarly,

\[ ds_y = \hat{y} \, dx \, dz \quad (x-z \text{ plane}), \quad (3.35b) \]

\[ ds_z = \hat{z} \, dx \, dy \quad (x-y \text{ plane}). \quad (3.35c) \]

A differential volume equals the product of all three differential lengths:

\[ dV = dx \, dy \, dz. \quad (3.36) \]
Table 3-1: Summary of vector relations.

<table>
<thead>
<tr>
<th>Coordinate variables</th>
<th>Cartesian Coordinates</th>
<th>Cylindrical Coordinates</th>
<th>Spherical Coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vector representation $\mathbf{A} =$</td>
<td>$\hat{x} A_x + \hat{y} A_y + \hat{z} A_z$</td>
<td>$\hat{r} A_r + \hat{\phi} A_\phi + \hat{z} A_z$</td>
<td>$\hat{R} A_R + \hat{\theta} A_\theta + \hat{\phi} A_\phi$</td>
</tr>
<tr>
<td>Magnitude of $\mathbf{A}$ $</td>
<td>A</td>
<td>=$</td>
<td>$\sqrt{A_x^2 + A_y^2 + A_z^2}$</td>
</tr>
<tr>
<td>Position vector $\overrightarrow{OP}$</td>
<td>$\hat{x} x_1 + \hat{y} y_1 + \hat{z} z_1$, for $P = (x_1, y_1, z_1)$</td>
<td>$\hat{r} r_1 + \hat{\phi} \phi_1 + \hat{z} z_1$, for $P = (r_1, \phi_1, z_1)$</td>
<td>$\hat{R} R_1$, for $P = (R_1, \theta_1, \phi_1)$</td>
</tr>
<tr>
<td>Base vectors properties</td>
<td>$\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1$</td>
<td>$\hat{r} \cdot \hat{r} = \hat{\phi} \cdot \hat{\phi} = \hat{z} \cdot \hat{z} = 1$</td>
<td>$\hat{R} \cdot \hat{R} = \hat{\theta} \cdot \hat{\theta} = \hat{\phi} \cdot \hat{\phi} = 1$</td>
</tr>
<tr>
<td></td>
<td>$\hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{x} = 0$</td>
<td>$\hat{r} \cdot \hat{\phi} = \hat{\phi} \cdot \hat{z} = \hat{z} \cdot \hat{r} = 0$</td>
<td>$\hat{R} \cdot \hat{\theta} = \hat{\theta} \cdot \hat{\phi} = \hat{\phi} \cdot \hat{R} = 0$</td>
</tr>
<tr>
<td></td>
<td>$\hat{x} \times \hat{y} = \hat{z}$</td>
<td>$\hat{r} \times \hat{\phi} = \hat{z}$</td>
<td>$\hat{R} \times \hat{\theta} = \hat{\phi}$</td>
</tr>
<tr>
<td></td>
<td>$\hat{y} \times \hat{z} = \hat{x}$</td>
<td>$\hat{\phi} \times \hat{z} = \hat{r}$</td>
<td>$\hat{\theta} \times \hat{R} = \hat{\phi}$</td>
</tr>
<tr>
<td></td>
<td>$\hat{z} \times \hat{x} = \hat{y}$</td>
<td>$\hat{z} \times \hat{r} = \hat{\phi}$</td>
<td>$\hat{\phi} \times \hat{R} = \hat{\theta}$</td>
</tr>
<tr>
<td>Dot product $\mathbf{A} \cdot \mathbf{B} =$</td>
<td>$A_x B_x + A_y B_y + A_z B_z$</td>
<td>$A_r B_r + A_\phi B_\phi + A_z B_z$</td>
<td>$A_R B_R + A_\theta B_\theta + A_\phi B_\phi$</td>
</tr>
<tr>
<td>Cross product $\mathbf{A} \times \mathbf{B} =$</td>
<td>$\begin{vmatrix} \hat{x} &amp; \hat{y} &amp; \hat{z} \ A_x &amp; A_y &amp; A_z \ B_x &amp; B_y &amp; B_z \end{vmatrix}$</td>
<td>$\begin{vmatrix} \hat{r} &amp; \hat{\phi} &amp; \hat{z} \ A_r &amp; A_\phi &amp; A_z \ B_r &amp; B_\phi &amp; B_z \end{vmatrix}$</td>
<td>$\begin{vmatrix} \hat{R} &amp; \hat{\theta} &amp; \hat{\phi} \ A_R &amp; A_\theta &amp; A_\phi \ B_R &amp; B_\theta &amp; B_\phi \end{vmatrix}$</td>
</tr>
<tr>
<td>Differential length $d\ell =$</td>
<td>$\hat{x} , dx + \hat{y} , dy + \hat{z} , dz$</td>
<td>$\hat{r} , dr + \hat{\phi} , d\phi + \hat{z} , dz$</td>
<td>$\hat{R} , dR + \hat{\theta} , d\theta + \hat{\phi} , d\phi$</td>
</tr>
<tr>
<td>Differential surface areas</td>
<td>$ds_x = \hat{x} , dy , dz$</td>
<td>$ds_r = \hat{r} , d\phi , dz$</td>
<td>$ds_R = \hat{R} , R^2 \sin \theta , d\theta , d\phi$</td>
</tr>
<tr>
<td></td>
<td>$ds_y = \hat{y} , dx , dz$</td>
<td>$ds_{\phi} = \hat{\phi} , dr , dz$</td>
<td>$ds_{\theta} = \hat{\theta} , R , \sin \theta , dR , d\phi$</td>
</tr>
<tr>
<td></td>
<td>$ds_z = \hat{z} , dx , dy$</td>
<td>$ds_z = \hat{z} , dr , d\phi$</td>
<td>$ds_{\phi} = \hat{\phi} , R , dR , d\theta$</td>
</tr>
<tr>
<td>Differential volume $dV =$</td>
<td>$dx , dy , dz$</td>
<td>$r , dr , d\phi , dz$</td>
<td>$R^2 \sin \theta , dR , d\theta , d\phi$</td>
</tr>
</tbody>
</table>
Cylindrical Coordinate System

The position vector \( \overrightarrow{OP} \) shown in Fig. 3-9 has components along \( r \) and \( z \) only. Thus,

\[
\mathbf{R}_1 = \overrightarrow{OP} = \hat{r}r_1 + \hat{z}z_1.
\]  
(3.40)

The mutually perpendicular base vectors are \( \hat{r}, \hat{\phi}, \) and \( \hat{z} \), with \( \hat{r} \) pointing away from the origin along \( r \), \( \hat{\phi} \) pointing in a direction tangential to the cylindrical surface, and \( \hat{z} \) pointing along the vertical. Unlike the Cartesian system, in which the base vectors \( \hat{x}, \hat{y}, \) and \( \hat{z} \) are independent of the location of \( P \), in the cylindrical system both \( \hat{r} \) and \( \hat{\phi} \) are functions of \( \phi \).
Cylindrical Coordinate System

The base unit vectors obey the following right-hand cyclic relations:

\[ \hat{r} \times \hat{\phi} = \hat{z}, \quad \hat{\phi} \times \hat{z} = \hat{r}, \quad \hat{z} \times \hat{r} = \hat{\phi}, \quad (3.37) \]

and like all unit vectors, \( \hat{r} \cdot \hat{r} = \hat{\phi} \cdot \hat{\phi} = \hat{z} \cdot \hat{z} = 1 \), and \( \hat{r} \times \hat{r} = \hat{\phi} \times \hat{\phi} = \hat{z} \times \hat{z} = 0 \).

In cylindrical coordinates, a vector is expressed as

\[ \mathbf{A} = \hat{a} |\mathbf{A}| = \hat{r} A_r + \hat{\phi} A_\phi + \hat{z} A_z, \quad (3.38) \]

\[ dl_r = dr, \quad dl_\phi = r \, d\phi, \quad dl_z = dz. \quad (3.41) \]

Note that the differential length along \( \hat{\phi} \) is \( r \, d\phi \), not just \( d\phi \). The differential length \( dl \) in cylindrical coordinates is given by

\[ dl = \hat{r} \, dl_r + \hat{\phi} \, dl_\phi + \hat{z} \, dl_z = \hat{r} \, dr + \hat{\phi} r \, d\phi + \hat{z} \, dz. \quad (3.42) \]

Figure 3-10: Differential areas and volume in cylindrical coordinates.
Example 3-3: Distance Vector in Cylindrical Coordinates

Find an expression for the unit vector of vector \( \mathbf{A} \) shown in Fig. 3-11 in cylindrical coordinates.

**Solution:** In triangle \( OP_1P_2 \),

\[
\overrightarrow{OP_2} = \overrightarrow{OP_1} + \mathbf{A}.
\]

Hence,

\[
\mathbf{A} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \hat{r}r_0 - \hat{z}h,
\]

and

\[
\hat{a} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\hat{r}r_0 - \hat{z}h}{\sqrt{r_0^2 + h^2}}.
\]

We note that the expression for \( \mathbf{A} \) is independent of \( \phi_0 \). That is, all vectors from point \( P_1 \) to any point on the circle defined by \( r = r_0 \) in the \( x-y \) plane are equal in the cylindrical coordinate system. The ambiguity can be eliminated by specifying that \( \mathbf{A} \) passes through a point whose \( \phi = \phi_0 \).
Example 3-4: Cylindrical Area

Find the area of a cylindrical surface described by \( r = 5 \), \( 30^\circ \leq \phi \leq 60^\circ \), and \( 0 \leq z \leq 3 \) (Fig. 3-12).

Solution: The prescribed surface is shown in Fig. 3-12. Use of Eq. (3.43a) for a surface element with constant \( r \) gives

\[
S = r \int_{\phi=30^\circ}^{60^\circ} d\phi \int_{z=0}^{3} d\ z
\]

\[
= 5\phi \left|_{\pi/6}^{\pi/3} \right|_0^3
\]

\[
= \frac{5\pi}{2}.
\]

Note that \( \phi \) had to be converted to radians before evaluating the integration limits.

Figure 3-12: Cylindrical surface of Example 3-4.
Cylindrical Coordinate System

The base unit vectors obey the following right-hand cyclic relations:

\[ \hat{r} \times \hat{\phi} = \hat{z}, \quad \hat{\phi} \times \hat{z} = \hat{r}, \quad \hat{z} \times \hat{r} = \hat{\phi}, \]

and like all unit vectors, \( \hat{r} \cdot \hat{r} = \hat{\phi} \cdot \hat{\phi} = \hat{z} \cdot \hat{z} = 1 \), and \( \hat{r} \times \hat{r} = \hat{\phi} \times \hat{\phi} = \hat{z} \times \hat{z} = 0 \).

In cylindrical coordinates, a vector is expressed as

\[ \mathbf{A} = \hat{a}|\mathbf{A}| = \hat{r}A_r + \hat{\phi}A_\phi + \hat{z}A_z, \]

\[ dl_r = dr, \quad dl_\phi = r\,d\phi, \quad dl_z = dz. \]

Note that the differential length along \( \hat{\phi} \) is \( r\,d\phi \), not just \( d\phi \). The differential length \( dl \) in cylindrical coordinates is given by

\[ dl = \hat{r}\,dl_r + \hat{\phi}\,dl_\phi + \hat{z}\,dl_z = \hat{r}\,dr + \hat{\phi}r\,d\phi + \hat{z}\,dz. \]
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Solution: The prescribed surface is shown in Fig. 3-12. Use of Eq. (3.43a) for a surface element with constant \( r \) gives

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= \frac{5\pi}{2}.
\]

Note that \( \phi \) had to be converted to radians before evaluating the integration limits.
CD Module 3.1 Points and Vectors  Examine the relationships between Cartesian coordinates \((x, y)\) and cylindrical coordinates \((r, \phi)\) for points and vectors.
Spherical Coordinate System

\[ \hat{R} \times \hat{\theta} = \hat{\phi}, \quad \hat{\theta} \times \hat{\phi} = \hat{R}, \quad \hat{\phi} \times \hat{R} = \hat{\theta}. \quad (3.45) \]

A vector with components \( A_R, A_\theta, \) and \( A_\phi \) is written as

\[ \mathbf{A} = \hat{a}|\mathbf{A}| = \hat{R}A_R + \hat{\theta}A_\theta + \hat{\phi}A_\phi, \quad (3.46) \]

and its magnitude is

\[ |\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \sqrt{A_R^2 + A_\theta^2 + A_\phi^2}. \quad (3.47) \]

The position vector of point \( P = (R_1, \theta_1, \phi_1) \) is simply

\[ \mathbf{R}_1 = \overrightarrow{OP} = \hat{R}R_1, \quad (3.48) \]
Example 3-5: Surface Area in Spherical Coordinates

The spherical strip shown in Fig. 3-15 is a section of a sphere of radius 3 cm. Find the area of the strip.

![Figure 3-15: Spherical strip of Example 3-5.](image)

Solution: Use of Eq. (3.50b) for the area of an elemental spherical area with constant radius $R$ gives

$$ S = R^2 \int_{0}^{2\pi} \sin \theta \, d\theta \int_{0}^{\theta} d\phi $$

$$ = 9(-\cos \theta) \int_{30^\circ}^{60^\circ} \int_{0}^{2\pi} \sin \theta \, d\theta \, d\phi $$

$$ = 18\pi (\cos 30^\circ - \cos 60^\circ) = 20.7 \text{ cm}^2. $$

Example 3-6: Charge in a Sphere

A sphere of radius 2 cm contains a volume charge density $\rho_v$ given by

$$ \rho_v = 4 \cos^2 \theta \quad (\text{C/m}^3). $$

Find the total charge $Q$ contained in the sphere.

Solution:

$$ Q = \iiint_V \rho_v \, dV $$

$$ = \int_0^{2\pi} \int_0^{\pi} \int_0^{2\times10^{-2}} (4 \cos^2 \theta) R^2 \sin \theta \, dR \, d\theta \, d\phi $$

$$ = 4 \int_0^{2\pi} \int_0^{\pi} \left( \frac{R^3}{3} \right) \left[ \frac{2\times10^{-2}}{2} \right] \sin \theta \cos^2 \theta \, d\theta \, d\phi $$

$$ = \frac{32}{3} \times 10^{-6} \int_0^{\theta} \left( -\frac{\cos^3 \theta}{3} \right) \left[ 0 \right] d\phi $$

$$ = \frac{64}{9} \times 10^{-6} \int_0^{2\pi} d\phi $$

$$ = \frac{128\pi}{9} \times 10^{-6} \approx 44.68 \quad (\mu\text{C}). $$
# Table 3-1: Summary of vector relations.

<table>
<thead>
<tr>
<th>Coordinate variables</th>
<th>Cartesian Coordinates</th>
<th>Cylindrical Coordinates</th>
<th>Spherical Coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coordinate variables</td>
<td>$x, y, z$</td>
<td>$r, \phi, z$</td>
<td>$R, \theta, \phi$</td>
</tr>
<tr>
<td>Vector representation $\mathbf{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$</td>
<td>$\hat{r}A_r + \hat{\phi}A_\phi + \hat{z}A_z$</td>
<td>$\hat{R}A_R + \hat{\theta}A_\theta + \hat{\phi}A_\phi$</td>
<td></td>
</tr>
<tr>
<td>Magnitude of $\mathbf{A}$ $</td>
<td>A</td>
<td>= \sqrt{A_x^2 + A_y^2 + A_z^2}$</td>
<td>$\sqrt{A_r^2 + A_\phi^2 + A_z^2}$</td>
</tr>
<tr>
<td>Position vector $\overrightarrow{OP_1} = \hat{x}x_1 + \hat{y}y_1 + \hat{z}z_1$, for $P = (x_1, y_1, z_1)$</td>
<td>$\hat{r}r_1 + \hat{z}z_1$, for $P = (r_1, \phi_1, z_1)$</td>
<td>$\hat{R}R_1$, for $P = (R_1, \theta_1, \phi_1)$</td>
<td></td>
</tr>
<tr>
<td>Base vectors properties</td>
<td>$\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1$</td>
<td>$\hat{r} \cdot \hat{r} = \hat{\phi} \cdot \hat{\phi} = \hat{z} \cdot \hat{z} = 1$</td>
<td>$\hat{R} \cdot \hat{R} = \hat{\theta} \cdot \hat{\theta} = \hat{\phi} \cdot \hat{\phi} = 1$</td>
</tr>
<tr>
<td></td>
<td>$\hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{x} = 0$</td>
<td>$\hat{r} \cdot \hat{\phi} = \hat{\phi} \cdot \hat{z} = \hat{z} \cdot \hat{r} = 0$</td>
<td>$\hat{R} \cdot \hat{\theta} = \hat{\theta} \cdot \hat{\phi} = \hat{\phi} \cdot \hat{R} = 0$</td>
</tr>
<tr>
<td></td>
<td>$\hat{x} \times \hat{y} = \hat{z}$</td>
<td>$\hat{r} \times \hat{\phi} = \hat{z}$</td>
<td>$\hat{R} \times \hat{\theta} = \hat{\phi}$</td>
</tr>
<tr>
<td></td>
<td>$\hat{y} \times \hat{z} = \hat{x}$</td>
<td>$\hat{\phi} \times \hat{z} = \hat{r}$</td>
<td>$\hat{\theta} \times \hat{\phi} = \hat{R}$</td>
</tr>
<tr>
<td></td>
<td>$\hat{z} \times \hat{x} = \hat{y}$</td>
<td>$\hat{z} \times \hat{r} = \hat{\phi}$</td>
<td>$\hat{\phi} \times \hat{R} = \hat{\theta}$</td>
</tr>
<tr>
<td>Dot product $\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$</td>
<td>$A_r B_r + A_\phi B_\phi + A_z B_z$</td>
<td>$A_R B_R + A_\theta B_\theta + A_\phi B_\phi$</td>
<td></td>
</tr>
<tr>
<td>Cross product $\mathbf{A} \times \mathbf{B} =$</td>
<td>$\begin{vmatrix} \hat{x} &amp; \hat{y} &amp; \hat{z} \ A_x &amp; A_y &amp; A_z \ B_x &amp; B_y &amp; B_z \end{vmatrix}$</td>
<td>$\begin{vmatrix} \hat{r} &amp; \hat{\phi} &amp; \hat{z} \ A_r &amp; A_\phi &amp; A_z \ B_r &amp; B_\phi &amp; B_z \end{vmatrix}$</td>
<td>$\begin{vmatrix} \hat{R} &amp; \hat{\theta} &amp; \hat{\phi} \ A_R &amp; A_\theta &amp; A_\phi \ B_R &amp; B_\theta &amp; B_\phi \end{vmatrix}$</td>
</tr>
<tr>
<td>Differential length $dl =$</td>
<td>$\hat{x} dx + \hat{y} dy + \hat{z} dz$</td>
<td>$\hat{r} dr + \hat{\phi} r d\phi + \hat{z} dz$</td>
<td>$\hat{R} dR + \hat{\theta} R d\theta + \hat{\phi} R \sin \theta d\phi$</td>
</tr>
<tr>
<td>Differential surface areas</td>
<td>$ds_x = \hat{x} dy dz$</td>
<td>$ds_r = \hat{r} r d\phi dz$</td>
<td>$ds_R = \hat{R} R^2 \sin \theta d\theta d\phi$</td>
</tr>
<tr>
<td></td>
<td>$ds_y = \hat{y} dx dz$</td>
<td>$ds_\phi = \hat{\phi} dr dz$</td>
<td>$ds_\theta = \hat{\theta} R \sin \theta dR d\phi$</td>
</tr>
<tr>
<td></td>
<td>$ds_z = \hat{z} dx dy$</td>
<td>$ds_z = \hat{z} r dr d\phi$</td>
<td>$ds_\phi = \hat{\phi} R dR d\theta$</td>
</tr>
<tr>
<td>Differential volume $dV =$</td>
<td>$dx dy dz$</td>
<td>$r dr d\phi dz$</td>
<td>$R^2 \sin \theta dR d\theta d\phi$</td>
</tr>
</tbody>
</table>
Spherical Coordinate System

\[ \hat{R} \times \hat{\theta} = \hat{\phi}, \quad \hat{\theta} \times \hat{\phi} = \hat{R}, \quad \hat{\phi} \times \hat{R} = \hat{\theta}. \] (3.45)

A vector with components \( A_R, A_\theta, \) and \( A_\phi \) is written as

\[ \mathbf{A} = \hat{a}|\mathbf{A}| = \hat{R}A_R + \hat{\theta}A_\theta + \hat{\phi}A_\phi, \] (3.46)

and its magnitude is

\[ |\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \sqrt{A_R^2 + A_\theta^2 + A_\phi^2}. \] (3.47)

The position vector of point \( P = (R_1, \theta_1, \phi_1) \) is simply

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The spherical strip shown in Fig. 3-15 is a section of a sphere of radius 3 cm. Find the area of the strip.

Solution: Use of Eq. (3.50b) for the area of an elemental spherical area with constant radius \( R \) gives

\[
S = R^2 \int_{\theta=30^\circ}^{60^\circ} \sin \theta \ d\theta \int_{\phi=0}^{2\pi} d\phi
\]

\[
= 9(-\cos \theta) \bigg|_{30^\circ}^{60^\circ} \phi \bigg|_{0}^{2\pi} \ (\text{cm}^2)
\]

\[
= 18\pi(\cos 30^\circ - \cos 60^\circ) = 20.7 \text{ cm}^2.
\]

Figure 3-15: Spherical strip of Example 3-5.

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\[
\rho_v = 4\cos^2 \theta \quad (\text{C/m}^3).
\]

Find the total charge \( Q \) contained in the sphere.

Solution:

\[
Q = \int_V \rho_v \ dV
\]

\[
= \int_0^{2\pi} \int_0^{\pi} \int_0^{2\times10^{-2}} (4\cos^2 \theta) R^2 \sin \theta \ dR \ d\theta \ d\phi
\]

\[
= 4 \int_0^{2\pi} \int_0^{\pi} \left( \frac{R^3}{3} \right) \sin \theta \cos^2 \theta \ d\theta \ d\phi
\]

\[
= \frac{32}{3} \times 10^{-6} \int_0^{2\pi} \left( -\frac{\cos^3 \theta}{3} \right) \bigg|_0^{\pi} \ d\phi
\]

\[
= \frac{64}{9} \times 10^{-6} \int_0^{2\pi} d\phi
\]

\[
= \frac{128\pi}{9} \times 10^{-6} = 44.68 \ (\mu\text{C}).
\]
How does a GPS receiver determine its location?
GPS: Minimum of 4 Satellites Needed

Unknown: location of receiver \((x_0, y_0, z_0)\)
Also unknown: time offset of receiver clock \(t_0\)

Quantities known with high precision:
locations of satellites and their atomic clocks (satellites use expensive high precision clocks, whereas receivers do not)

Solving for 4 unknowns requires at least 4 equations (four satellites)

\[
\begin{align*}
  d_1^2 &= (x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 = c \left[(t_1 + t_0)\right]^2, \\
  d_2^2 &= (x_2 - x_0)^2 + (y_2 - y_0)^2 + (z_2 - z_0)^2 = c \left[(t_2 + t_0)\right]^2, \\
  d_3^2 &= (x_3 - x_0)^2 + (y_3 - y_0)^2 + (z_3 - z_0)^2 = c \left[(t_3 + t_0)\right]^2, \\
  d_4^2 &= (x_4 - x_0)^2 + (y_4 - y_0)^2 + (z_4 - z_0)^2 = c \left[(t_4 + t_0)\right]^2.
\end{align*}
\]

Figure TF5-3: Automobile GPS receiver at location \((x_0, y_0, z_0)\).
Coordinate Transformations: Coordinates

- To solve a problem, we select the coordinate system that best fits its geometry.
- Sometimes we need to transform between coordinate systems.

\[
\begin{align*}
    r &= \sqrt{x^2 + y^2}, \\
    \phi &= \tan^{-1} \left( \frac{y}{x} \right),
\end{align*}
\]

and the inverse relations are

\[
\begin{align*}
    x &= r \cos \phi, \\
    y &= r \sin \phi.
\end{align*}
\]

*Figure 3-16: Interrelationships between Cartesian coordinates \((x, y, z)\) and cylindrical coordinates \((r, \phi, z)\).*
Coordinate Transformations: Unit Vectors

\[
\hat{\mathbf{r}} \cdot \hat{\mathbf{x}} = \cos \phi, \quad \hat{\mathbf{r}} \cdot \hat{\mathbf{y}} = \sin \phi, \\
\hat{\phi} \cdot \hat{\mathbf{x}} = -\sin \phi, \quad \hat{\phi} \cdot \hat{\mathbf{y}} = \cos \phi.
\]

\[
\hat{\mathbf{r}} = \hat{\mathbf{x}} \cos \phi + \hat{\mathbf{y}} \sin \phi. \\
\hat{\phi} = -\hat{\mathbf{x}} \sin \phi + \hat{\mathbf{y}} \cos \phi.
\]

\[
\hat{\mathbf{x}} = \hat{\mathbf{r}} \cos \phi - \hat{\phi} \sin \phi, \\
\hat{\mathbf{y}} = \hat{\mathbf{r}} \sin \phi + \hat{\phi} \cos \phi.
\]
Table 3-2: Coordinate transformation relations.

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Coordinate Variables</th>
<th>Unit Vectors</th>
<th>Vector Components</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cartesian to cylindrical</strong></td>
<td>( r = \sqrt{x^2 + y^2} ) [ \phi = \tan^{-1}(y/x) ] [ z = z ]</td>
<td>( \hat{r} = \hat{x} \cos \phi + \hat{y} \sin \phi ) [ \hat{\phi} = -\hat{x} \sin \phi + \hat{y} \cos \phi ] [ \hat{z} = \hat{z} ]</td>
<td>( A_r = A_x \cos \phi + A_y \sin \phi ) [ A_\phi = -A_x \sin \phi + A_y \cos \phi ] [ A_z = A_z ]</td>
</tr>
<tr>
<td><strong>Cylindrical to Cartesian</strong></td>
<td>( x = r \cos \phi ) [ y = r \sin \phi ] [ z = z ]</td>
<td>( \hat{x} = \hat{r} \cos \phi - \hat{\phi} \sin \phi ) [ \hat{y} = \hat{r} \sin \phi + \hat{\phi} \cos \phi ] [ \hat{z} = \hat{z} ]</td>
<td>( A_x = A_r \cos \phi - A_\phi \sin \phi ) [ A_y = A_r \sin \phi + A_\phi \cos \phi ] [ A_z = A_z ]</td>
</tr>
<tr>
<td><strong>Cartesian to spherical</strong></td>
<td>( R = \sqrt{x^2 + y^2 + z^2} ) [ \theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right) ] [ \phi = \tan^{-1}(y/x) ]</td>
<td>( \hat{R} = \hat{x} \sin \theta \cos \phi ) [ + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta ] [ \hat{\theta} = \hat{x} \cos \theta \cos \phi ] [ + \hat{y} \cos \theta \sin \phi - \hat{z} \sin \theta ] [ \hat{\phi} = -\hat{x} \sin \phi + \hat{y} \cos \phi ]</td>
<td>( A_R = A_x \sin \theta \cos \phi ) [ + A_y \sin \theta \sin \phi + A_z \cos \theta ] [ A_\theta = A_x \cos \theta \cos \phi ] [ + A_y \cos \theta \sin \phi - A_z \sin \theta ] [ A_\phi = -A_x \sin \phi + A_y \cos \phi ]</td>
</tr>
<tr>
<td><strong>Spherical to Cartesian</strong></td>
<td>( x = R \sin \theta \cos \phi ) [ y = R \sin \theta \sin \phi ] [ z = R \cos \theta ]</td>
<td>( \hat{x} = \hat{R} \sin \theta \cos \phi ) [ + \hat{\theta} \cos \theta \cos \phi - \hat{\phi} \sin \phi ] [ \hat{y} = \hat{R} \sin \theta \sin \phi ] [ + \hat{\theta} \cos \theta \sin \phi + \hat{\phi} \cos \phi ] [ \hat{z} = \hat{R} \cos \theta - \hat{\theta} \sin \theta ]</td>
<td>( A_x = A_R \sin \theta \cos \phi ) [ + A_\theta \cos \theta \cos \phi - A_\phi \sin \phi ] [ A_y = A_R \sin \theta \sin \phi ] [ + A_\theta \cos \theta \sin \phi + A_\phi \cos \phi ] [ A_z = A_R \cos \theta - A_\theta \sin \theta ]</td>
</tr>
<tr>
<td><strong>Cylindrical to spherical</strong></td>
<td>( R = \sqrt{r^2 + z^2} ) [ \theta = \tan^{-1}(r/z) ] [ \phi = \phi ]</td>
<td>( \hat{R} = \hat{r} \sin \theta + \hat{z} \cos \theta ) [ \hat{\theta} = \hat{r} \cos \theta - \hat{z} \sin \theta ] [ \hat{\phi} = \hat{\phi} ]</td>
<td>( A_R = A_r \sin \theta + A_z \cos \theta ) [ A_\theta = A_r \cos \theta - A_z \sin \theta ] [ A_\phi = A_\phi ]</td>
</tr>
<tr>
<td><strong>Spherical to cylindrical</strong></td>
<td>( r = R \sin \theta ) [ \phi = \phi ] [ z = R \cos \theta ]</td>
<td>( \hat{r} = \hat{R} \sin \theta + \hat{\theta} \cos \theta ) [ \hat{\phi} = \hat{\phi} ] [ \hat{z} = \hat{R} \cos \theta - \hat{\theta} \sin \theta ]</td>
<td>( A_r = A_R \sin \theta + A_\theta \cos \theta ) [ A_\phi = A_\phi ] [ A_z = A_R \cos \theta - A_\theta \sin \theta ]</td>
</tr>
</tbody>
</table>
Example 3-7: Cartesian to Cylindrical Transformations

Given point \( P_1 = (3, -4, 3) \) and vector \( \mathbf{A} = \hat{x}2 - \hat{y}3 + \hat{z}4 \), defined in Cartesian coordinates, express \( P_1 \) and \( \mathbf{A} \) in cylindrical coordinates and evaluate \( \mathbf{A} \) at \( P_1 \).

Solution: For point \( P_1 \), \( x = 3 \), \( y = -4 \), and \( z = 3 \). Using Eq. (3.51), we have

\[
r = \sqrt{x^2 + y^2} = 5, \quad \phi = \tan^{-1} \frac{y}{x} = -53.1^\circ = 306.9^\circ,
\]
and \( z \) remains unchanged. Hence, \( P_1 = (5, 306.9^\circ, 3) \) in cylindrical coordinates.

The cylindrical components of vector \( \mathbf{A} = \hat{r}A_r + \hat{\phi}A_{\phi} + \hat{z}A_z \) can be determined by applying Eqs. (3.58a) and (3.58b):

\[
A_r = A_x \cos \phi + A_y \sin \phi = 2 \cos \phi - 3 \sin \phi,
\]
\[
A_{\phi} = -A_x \sin \phi + A_y \cos \phi = -2 \sin \phi - 3 \cos \phi,
\]
\[
A_z = 4.
\]

Hence,

\[
\mathbf{A} = \hat{r}(2 \cos \phi - 3 \sin \phi) - \hat{\phi}(2 \sin \phi + 3 \cos \phi) + \hat{z}4.
\]

At point \( P \), \( \phi = 306.9^\circ \), which gives

\[
\mathbf{A} = \hat{r}3.60 - \hat{\phi}0.20 + \hat{z}4.
\]
Spherical Coordinate System

\[ \hat{R} \times \hat{\theta} = \hat{\phi}, \quad \hat{\theta} \times \hat{\phi} = \hat{R}, \quad \hat{\phi} \times \hat{R} = \hat{\theta}. \tag{3.45} \]

A vector with components \( A_R, A_\theta, \) and \( A_\phi \) is written as

\[ \mathbf{A} = \hat{a}|\mathbf{A}| = \hat{R}A_R + \hat{\theta}A_\theta + \hat{\phi}A_\phi, \tag{3.46} \]

and its magnitude is

\[ |\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \sqrt{A_R^2 + A_\theta^2 + A_\phi^2}. \tag{3.47} \]

The position vector of point \( P = (R_1, \theta_1, \phi_1) \) is simply

\[ \mathbf{R}_1 = \overrightarrow{OP} = \hat{R}R_1, \tag{3.48} \]
Example 3-8: Cartesian to Spherical Transformation

Express vector $\mathbf{A} = \hat{x}(x + y) + \hat{y}(y - x) + \hat{z}z$ in spherical coordinates.

**Solution:** Using the transformation relation for $A_R$ given in Table 3.2, we have

$$A_R = A_x \sin \theta \cos \phi + A_y \sin \theta \sin \phi + A_z \cos \theta$$
$$= (x + y) \sin \theta \cos \phi + (y - x) \sin \theta \sin \phi + z \cos \theta.$$

Using the expressions for $x, y,$ and $z$ given by Eq. (3.61c), we have

$$A_R = (R \sin \theta \cos \phi + R \sin \theta \sin \phi) \sin \theta \cos \phi$$
$$+ (R \sin \theta \sin \phi - R \sin \theta \cos \phi) \sin \theta \sin \phi + R \cos^2 \theta$$
$$= R \sin^2 \theta \cos^2 \phi + R \sin^2 \phi + R \cos^2 \theta$$
$$= R \sin^2 \theta + R \cos^2 \theta = R.$$

Similarly,

$$A_\theta = (x + y) \cos \theta \cos \phi + (y - x) \cos \theta \sin \phi - z \sin \theta,$$
$$A_\phi = -(x + y) \sin \phi + (y - x) \cos \phi,$$

and following the procedure used with $A_R$, we obtain

$$A_\theta = 0,$$
$$A_\phi = -R \sin \theta.$$

Hence,

$$\mathbf{A} = \hat{R}A_R + \hat{\theta}A_\theta + \hat{\phi}A_\phi = \hat{R}R - \hat{\phi}R \sin \theta.$$
Distance Between 2 Points

\[ d = |\mathbf{R}_{12}| \]
\[ = \left[ (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \right]^{1/2}. \]  \hspace{1cm} (3.66)

\[ d = \left[ (r_2 \cos \phi_2 - r_1 \cos \phi_1)^2 \right. \]
\[ + (r_2 \sin \phi_2 - r_1 \sin \phi_1)^2 + (z_2 - z_1)^2 \right]^{1/2} \]
\[ = \left[ r_2^2 + r_1^2 - 2r_1 r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2 \right]^{1/2} \]
\[ \text{(cylindrical).} \] \hspace{1cm} (3.67)

\[ d = \left\{ R_2^2 + R_1^2 - 2 R_1 R_2 \cos \theta_2 \cos \theta_1 \right. \]
\[ + \sin \theta_1 \sin \theta_2 \cos(\phi_2 - \phi_1) \left. \right\}^{1/2} \]
\[ \text{(spherical).} \] \hspace{1cm} (3.68)
So far our math review has dealt with vector algebra and geometry. Now we want to review some vector calculus—particularly quantities needed to calculate EM fields, potentials, etc.

In vector calculus, we use three fundamental "operators" to describe the spatial variations of both scalar and vector. Today, we will discuss these "operators" or operators. One of these "operators" operates on scalar fields, the other two operate on vector fields.

<remember, a "field" just defines some "property" varying in space>
An example of a "scalar field" is one where temperature depends upon position in 3D space.

For example, on a weather map of US, we often see "isoline" of most. temperature - like going from Lake Tahoe to Santa Cruz. temp lower in Tahoe than SC

so for example

$T_1 < T_2 < T_3$

the change in $T$ across with respect to 3D coordinates ($s, x, y, z$) can be described as a partial derivative of $T$ with respect to the 3 coordinates

Santa Cruz
Gradient of A Scalar Field

**Figure 3-19:** Differential distance vector $dl$ between points $P_1$ and $P_2$.

From differential calculus, the temperature difference between points $P_1$ and $P_2$, $dT = T_2 - T_1$, is

$$dT = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz. \quad (3.71)$$

Because $dx = \hat{x} \cdot dl$, $dy = \hat{y} \cdot dl$, and $dz = \hat{z} \cdot dl$, Eq. (3.71) can be rewritten as

$$dT = \hat{x} \frac{\partial T}{\partial x} \cdot dl + \hat{y} \frac{\partial T}{\partial y} \cdot dl + \hat{z} \frac{\partial T}{\partial z} \cdot dl$$

$$= \left[ \hat{x} \frac{\partial T}{\partial x} + \hat{y} \frac{\partial T}{\partial y} + \hat{z} \frac{\partial T}{\partial z} \right] \cdot dl. \quad (3.72)$$

$$\nabla T = \text{grad } T = \hat{x} \frac{\partial T}{\partial x} + \hat{y} \frac{\partial T}{\partial y} + \hat{z} \frac{\partial T}{\partial z}. \quad (3.72)$$

Equation (3.71) can then be expressed as

$$dT = \nabla T \cdot dl. \quad (3.73)$$

The symbol $\nabla$ is called the del or gradient operator and is defined as

$$\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \quad \text{(Cartesian).} \quad (3.74)$$
NOTE: $\nabla$ has no physical meaning itself.

BUT when $\nabla$ "operates" on a physical scalar quantity, the result is a vector whose magnitude equals the maximum rate of change of that quantity per unit distance and whose direction is along the direction of max. increase.

We express $dT = \nabla T \cdot dr$ (i)
Gradient (cont.)

With \( dl = \hat{a}_l \, dl \), where \( \hat{a}_l \) is the unit vector of \( dl \), the directional derivative of \( T \) along \( \hat{a}_l \) is

\[
\frac{dT}{dl} = \nabla T \cdot \hat{a}_l. \quad (3.75)
\]

We can find the difference \((T_2 - T_1)\), where \( T_1 = T(x_1, y_1, z_1) \) and \( T_2 = T(x_2, y_2, z_2) \) are the values of \( T \) at points \( P_1 = (x_1, y_1, z_1) \) and \( P_2 = (x_2, y_2, z_2) \) not necessarily infinitesimally close to one another, by integrating both sides of Eq. (3.73). Thus,

\[
T_2 - T_1 = \int_{P_1}^{P_2} \nabla T \cdot dl. \quad (3.76)
\]

Example 3-9: Directional Derivative

Find the directional derivative of \( T = x^2 + y^2z \) along direction \( \hat{x}2 + \hat{y}3 - \hat{z}2 \) and evaluate it at \((1, -1, 2)\).

Solution: First, we find the gradient of \( T \):

\[
\nabla T = \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) (x^2 + y^2z)
\]

\[
= \hat{x}2x + \hat{y}2yz + \hat{z}y^2.
\]

We denote \( l \) as the given direction,

\[
l = \hat{x}2 + \hat{y}3 - \hat{z}2.
\]

Its unit vector is

\[
\hat{a}_l = \frac{l}{||l||} = \frac{\hat{x}2 + \hat{y}3 - \hat{z}2}{\sqrt{2^2 + 3^2 + 2^2}} = \frac{\hat{x}2 + \hat{y}3 - \hat{z}2}{\sqrt{17}}.
\]

Application of Eq. (3.75) gives

\[
\frac{dT}{dl} = \nabla T \cdot \hat{a}_l = (\hat{x}2x + \hat{y}2yz + \hat{z}y^2) \cdot \left( \frac{\hat{x}2 + \hat{y}3 - \hat{z}2}{\sqrt{17}} \right)
\]

\[
= \frac{4x + 6yz - 2y^2}{\sqrt{17}}.
\]

At \((1, -1, 2)\),

\[
\frac{dT}{dl} \bigg|_{(1,-1,2)} = \frac{4 - 12 - 2}{\sqrt{17}} = \frac{-10}{\sqrt{17}}.
\]
CD Module 3.2 Gradient

Select a scalar function $f(x, y, z)$, evaluate its gradient, and display both in an appropriate 2-D plane.

\[
\nabla (\cos(x-y)) = \hat{x} \frac{d(\cos(x-y))}{dx} + \hat{y} \frac{d(\cos(x-y))}{dy} + \hat{z} \frac{d(\cos(x-y))}{dz}
\]

\[
= -\hat{x} \sin(x-y) + \hat{y} \sin(x-y)
\]

\[
= -\sin(x-y)(\hat{x} - \hat{y})
\]
Divergence of a Vector Field

At a surface boundary, flux density is defined as the amount of outward flux crossing a unit surface $ds$:

\[
\text{Flux density of } \mathbf{E} = \frac{\mathbf{E} \cdot ds}{|ds|} = \frac{\mathbf{E} \cdot \hat{n} \, ds}{ds} = \mathbf{E} \cdot \hat{n}, \tag{3.85}
\]

where $\hat{n}$ is the normal to $ds$. The total flux outwardly crossing a closed surface $S$, such as the enclosed surface of the imaginary sphere outlined in Fig. 3-20, is

\[
\text{Total flux} = \oint_{S} \mathbf{E} \cdot ds. \tag{3.86}
\]

From the definition of the divergence of $\mathbf{E}$ given by Eq. (3.95), field $\mathbf{E}$ has positive divergence if the net flux out of surface $S$ is positive, which may be "viewed" as if volume $\Delta V$ contains a source of field lines. If the divergence is negative, $\Delta V$ may be viewed as containing a sink of field lines because the net flux is into $\Delta V$. For a uniform field $\mathbf{E}$, the same amount of flux enters $\Delta V$ as leaves it; hence, its divergence is zero and the field is said to be divergenceless.

\[
\text{div } \mathbf{E} \triangleq \lim_{\Delta V \to 0} \frac{\int_{S} \mathbf{E} \cdot ds}{\Delta V}, \tag{3.95}
\]

where $S$ encloses the elemental volume $\Delta V$. Instead of denoting the divergence of $\mathbf{E}$ by div $\mathbf{E}$, it is common practice to denote it as $\nabla \cdot \mathbf{E}$. That is,

\[
\nabla \cdot \mathbf{E} = \text{div } \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \tag{3.96}
\]
Divergence Theorem

\[ \int_{V} \nabla \cdot \mathbf{E} \, dV = \oint_{S} \mathbf{E} \cdot d\mathbf{s} \]  
(divergence theorem).

(3.98)

Useful tool for converting integration over a volume to one over the surface enclosing that volume, and vice versa.
CD Module 3.3 Divergence
Select a vector function \( \mathbf{v}(x, y, z) \), evaluate its divergence, and display both in an appropriate 2-D plane.

Graphics Created with Wolfram Mathematica®
Maxwell’s Equations

**Table 6-1**: Maxwell’s equations.

<table>
<thead>
<tr>
<th>Reference</th>
<th>Differential Form</th>
<th>Integral Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauss’s law</td>
<td>$\nabla \cdot \mathbf{D} = \rho_v$</td>
<td>$\oint_S \mathbf{D} \cdot ds = Q$</td>
</tr>
<tr>
<td>Faraday’s law</td>
<td>$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$</td>
<td>$\oint_C \mathbf{E} \cdot dl = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot ds$</td>
</tr>
<tr>
<td>Gauss’s law for magnetism</td>
<td>$\nabla \cdot \mathbf{B} = 0$</td>
<td>$\oint_S \mathbf{B} \cdot ds = 0$</td>
</tr>
<tr>
<td>Ampère’s law</td>
<td>$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$</td>
<td>$\oint_C \mathbf{H} \cdot dl = \int_S \left( \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot ds$</td>
</tr>
</tbody>
</table>

*For a stationary surface $S$.**
Example 3-11: Calculating the Divergence

Determine the divergence of each of the following vector fields and then evaluate them at the indicated points:

(a) \( \mathbf{E} = \hat{x}3x^2 + \hat{y}2z + \hat{z}x^2z \) at \((2, -2, 0)\);

(b) \( \mathbf{E} = \hat{R}(a^3 \cos \theta/R^2) - \hat{\theta}(a^3 \sin \theta/R^2) \) at \((a/2, 0, \pi)\).

Solution:

(a) \[ \nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \]

\[ = \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(x^2z) \]

\[ = 6x + 0 + x^2 \]

\[ = x^2 + 6x. \]

At \((2, -2, 0)\), \( \nabla \cdot \mathbf{E} \bigg|_{(2,-2,0)} = 16. \)

(b) From the expression given on the inside of the back cover of the book for the divergence of a vector in spherical coordinates, it follows that

\[ \nabla \cdot \mathbf{E} = \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 E_R) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} (E_\theta \sin \theta) \]

\[ + \frac{1}{R \sin \theta} \frac{\partial E_\phi}{\partial \phi} \]

\[ = \frac{1}{R^2} \frac{\partial}{\partial R} (a^3 \cos \theta) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} \left( -\frac{a^3 \sin^2 \theta}{R^2} \right) \]

\[ = 0 - \frac{2a^3 \cos \theta}{R^3} \]

\[ = -\frac{2a^3 \cos \theta}{R^3}. \]

At \( R = a/2 \) and \( \theta = 0 \), \( \nabla \cdot \mathbf{E} \bigg|_{(a/2,0,\pi)} = -16. \)
Curl of a Vector Field

\[ \text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l}. \]

\[ \nabla \times \mathbf{B} = \text{curl } \mathbf{B} \]
\[ = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[ \hat{n} \oint_C \mathbf{B} \cdot d\mathbf{l} \right]_{\text{max}}. \quad (3.103) \]

Thus, curl \( \mathbf{B} \) is the circulation of \( \mathbf{B} \) per unit area, with the area \( \Delta s \) of the contour \( C \) being oriented such that the circulation is maximum.

Figure 3-22: Circulation is zero for the uniform field in (a), but it is not zero for the azimuthal field in (b).
Stokes’s Theorem

Stokes’s theorem converts the surface integral of the curl of a vector over an open surface $S$ into a line integral of the vector along the contour $C$ bounding the surface $S$.

For the geometry shown in Fig. 3-23, Stokes’s theorem states

$$
\int_{S} (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_{C} \mathbf{B} \cdot d\mathbf{l} \quad \text{(Stokes’s theorem),}
$$

(3.107)

![Diagram showing Stokes’s Theorem](image)

**Figure 3-23:** The direction of the unit vector $\mathbf{\hat{n}}$ is along the thumb when the other four fingers of the right hand follow $d\mathbf{l}$. 
CD Module 3.4 Curl Select a vector \( \mathbf{v}(x, y) \), evaluate its curl, and display both in the \( x-y \) plane.

Module 3.4 Curl

Input

Select a function: \( \mathbf{v}(x, y) = \hat{x}\sin(\pi y) + \hat{y}\sin(\pi x) \)

Function: \( \mathbf{v}(x, y) = \hat{x}\sin(\pi y) + \hat{y}\sin(\pi x) \)

\[ \nabla \times \mathbf{v}(x, y) = \hat{\mathbf{z}}(\pi \cos(\pi x) - \pi \cos(\pi y)) \]

- \( \hat{\mathbf{z}}(\pi \cos(\pi x) + \pi \cos(\pi y)) \)

- \( \mathbf{v}(x, y) \) (arrows) and \( f(x, y) \) (colors)
- \( \mathbf{v}(x, y) \) (lines) and \( f(x, y) \) (colors)

\[ \nabla \times \mathbf{v} = \hat{\mathbf{z}} f(x, y) \]

\[ f(x, y) = \frac{d(\sin(\pi x))}{dx} - \frac{d(\sin(\pi y))}{dy} = \pi \cos(\pi x) - \pi \cos(\pi y) \]
Tech Brief 6: X-Ray Computed Tomography

- For each anatomical slice, the CT scanner generates on the order of $7 \times 10^5$ measurements (1,000 angular orientations x 700 detector channels)
- Use of vector calculus allows the extraction of the 2-D image of a slice
- Combining multiple slices generates a 3-D scan

*Figure TF6-3: Basic elements of a CT scanner.*
Chapter 3 Relationships

Distance Between Two Points
\[ d = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2} \]
\[ d = [r_2^2 + r_1^2 - 2r_1r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2]^{1/2} \]
\[ d = \left\{ R_2^2 + R_1^2 - 2R_1 R_2 [\cos \theta_2 \cos \theta_1 + \sin \theta_1 \sin \theta_2 \cos(\phi_2 - \phi_1)] \right\}^{1/2} \]

Coordinate Systems Table 3-1
Coordinate Transformations Table 3-2

Vector Operators
\[ \nabla T = \hat{x} \frac{\partial T}{\partial x} + \hat{y} \frac{\partial T}{\partial y} + \hat{z} \frac{\partial T}{\partial z} \]
\[ \nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \]
\[ \nabla \times \mathbf{B} = \hat{x} \left( \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) + \hat{y} \left( \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) + \hat{z} \left( \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \]
\[ \nabla^2 \mathbf{V} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \]
(see back cover for cylindrical and spherical coordinates)

Stokes’s Theorem
\[ \int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_C \mathbf{B} \cdot d\mathbf{l} \]
Divergence Theorem

\[ \int_{V} \nabla \cdot \mathbf{E} \, dv = \oint_{S} \mathbf{E} \cdot ds \]  
(divergence theorem).

(3.98)

Useful tool for converting integration over a volume to one over the surface enclosing that volume, and vice versa.
Laplacian Operator

Laplacian of a Scalar Field

\[ \nabla^2 V = \nabla \cdot (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}. \tag{3.110} \]

Laplacian of a Vector Field

\[ \nabla^2 \mathbf{E} = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \mathbf{E} \]
\[ = \hat{x} \nabla^2 E_x + \hat{y} \nabla^2 E_y + \hat{z} \nabla^2 E_z \]

Useful Relation

\[ \nabla^2 \mathbf{E} = \nabla (\nabla \cdot \mathbf{E}) - \nabla \times (\nabla \times \mathbf{E}). \tag{3.113} \]
Chapter 3 Relationships

Distance Between Two Points

\[ d = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2} \]

\[ d = [r_2^2 + r_1^2 - 2r_1r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2]^{1/2} \]

\[ d = \left\{ R_2^2 + R_1^2 - 2R_1R_2[\cos \theta_2 \cos \theta_1 + \sin \theta_1 \sin \theta_2 \cos(\phi_2 - \phi_1)] \right\}^{1/2} \]

Coordinate Systems  Table 3-1
Coordinate Transformations  Table 3-2

Vector Products

\[ \mathbf{A} \cdot \mathbf{B} = AB \cos \theta_{AB} \]

\[ \mathbf{A} \times \mathbf{B} = \hat{n} AB \sin \theta_{AB} \]

\[ \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \]

\[ \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \]

Divergence Theorem

\[ \int \nabla \cdot \mathbf{E} \, dV = \oint_{S} \mathbf{E} \cdot ds \]

Vector Operators

\[ \nabla \mathbf{T} = \hat{x} \frac{\partial T}{\partial x} + \hat{y} \frac{\partial T}{\partial y} + \hat{z} \frac{\partial T}{\partial z} \]

\[ \nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \]

\[ \nabla \times \mathbf{B} = \hat{x} \left( \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) + \hat{y} \left( \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) + \hat{z} \left( \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \]

\[ \nabla^2 \mathbf{V} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \]

(see back cover for cylindrical and spherical coordinates)

Stokes’s Theorem

\[ \int_{S} (\nabla \times \mathbf{B}) \cdot ds = \oint_{C} \mathbf{B} \cdot d\mathbf{l} \]
First Preliminary Exam

TOPICS:

- The Electromagnetic spectrum
- Traveling and Standing waves
- Phasors and Complex Algebra
- Waves in lossless and lossy media
- Lumped element model of transmission lines
- Wave propagation in transmission lines
- Lossless transmission line
- Voltage and Reflection coefficients
- Standing wave ratio
- Wave impedance
- Characteristic impedance
- Power flow in transmission lines
- Transients in transmission lines
Preliminary exam format

5 problems
   Including
   One problem of short answers
       either a very short calculation
       or a knowledge question

   One sheet of notes
       (turned in with exam)

Hint: don’t try to make problems harder than they are. Take time to think about the question being asked.