EE 135, Winter 2012

Reading: finish Chapter 1, Ulaby et.al. 6th edition

Homework #1, due today, 1/19/12:

Homework #2: problems, due 1/26/12
  chap.1, 1.28
  chap.2, 2.1, 2.6, 2.7(first part),2.12, 2.13, 2.19

Lecture 4
EE 135. Winter 2012

Math Diagnostic Exam

If you received less than 50/100 score, Go through math refresher videos at The Khan Academy, www.khanacadamy.org <integral/differential calculus, differential equations, complex algebra, vector algebra>
Derivation of Wave Equations

\[-\frac{d\tilde{V}(z)}{dz} = (R' + j\omega L') \tilde{I}(z),\]
\[-\frac{d\tilde{I}(z)}{dz} = (G' + j\omega C') \tilde{V}(z).\]

Combining the two equations leads to:
\[\frac{d^2\tilde{V}(z)}{dz^2} - (R' + j\omega L')(G' + j\omega C') \tilde{V}(z) = 0,\]

Second-order differential equation

\[\gamma = \alpha + j\beta,\]

\[\alpha = \Re(\gamma) = \Re \left(\sqrt{(R' + j\omega L')(G' + j\omega C')}\right) \quad \text{(Np/m),} \quad (2.25a)\]
\[\beta = \Im(\gamma) = \Im \left(\sqrt{(R' + j\omega L')(G' + j\omega C')}\right) \quad \text{(rad/m).} \quad (2.25b)\]
Solution of Wave Equations (cont.)

Proposed form of solution:

\[
\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \quad \text{(V)},
\]

\[
\tilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \quad \text{(A)}. 
\]

Using:

\[
-\frac{d\tilde{V}(z)}{dz} = (R' + j\omega L') \tilde{I}(z),
\]

It follows that:

\[
\tilde{I}(z) = \frac{\gamma}{R' + j\omega L'} \left[ V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z} \right]
\]

Comparison of each term with the corresponding term in Eq. (2.26b) leads us to conclude that

\[
\frac{V_0^+}{I_0^+} = Z_0 = \frac{-V_0^-}{I_0^-},
\]

where

\[
Z_0 = \frac{R' + j\omega L'}{\gamma} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \quad \text{(\Omega)},
\]

is called the characteristic impedance of the line.
Solution of Wave Equations (cont.)

In general:

\[ V_0^+ = |V_0^+|e^{j\phi^+}, \]
\[ V_0^- = |V_0^-|e^{j\phi^-}. \]

\[
u(z, t) = \Re(\tilde{V}(z)e^{j\omega t}) = \Re \left[ \left( V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \right) e^{j\omega t} \right]
= \Re \left[ |V_0^+|e^{j\phi^+} e^{j\omega t} e^{-(\alpha+j\beta)z} + |V_0^-|e^{j\phi^-} e^{j\omega t} e^{(\alpha+j\beta)z} \right]
= |V_0^+|e^{-\alpha z} \cos(\omega t - \beta z + \phi^+) + |V_0^-|e^{\alpha z} \cos(\omega t + \beta z + \phi^-)\]

The presence of two waves on the line propagating in opposite directions produces a standing wave.

wave along +z because coefficients of t and z have opposite signs
wave along -z because coefficients of t and z have the same sign
Example 2-1: Air Line

An air line is a transmission line in which air separates the two conductors, which renders \( G' = 0 \) because \( \sigma = 0 \). In addition, assume that the conductors are made of a material with high conductivity so that \( R' \approx 0 \). For an air line with a characteristic impedance of 50 \( \Omega \) and a phase constant of 20 rad/m at 700 MHz, find the line inductance \( L' \) and the line capacitance \( C' \).

**Solution:** The following quantities are given:

\[
Z_0 = 50 \, \Omega, \quad \beta = 20 \, \text{rad/m},
\]

\[
f = 700 \, \text{MHz} = 7 \times 10^8 \, \text{Hz}.
\]

With \( R' = G' = 0 \), Eqs. (2.25b) and (2.29) reduce to

\[
\beta = \text{Im} \left[ \sqrt{(j\omega L')(j\omega C')} \right]
\]

\[
= \text{Im} \left( j\omega \sqrt{L'C'} \right) = \omega \sqrt{L'C'},
\]

\[
Z_0 = \sqrt{\frac{j\omega L'}{j\omega C'}} = \sqrt{\frac{L'}{C'}}.
\]

The ratio of \( \beta \) to \( Z_0 \) is

\[
\frac{\beta}{Z_0} = \omega C',
\]

or

\[
C' = \frac{\beta}{\omega Z_0} = \frac{20}{2\pi \times 7 \times 10^8 \times 50} = 9.09 \times 10^{-11} \, \text{F/m} = 90.9 \, \text{pF/m}.
\]

From \( Z_0 = \sqrt{L'/C'} \), it follows that

\[
L' = Z_0^2 C' = (50)^2 \times 90.9 \times 10^{-12} = 2.27 \times 10^{-7} \, \text{H/m} = 227 \, \text{nH/m}.
\]
CD Module 2.2 Coaxial Cable  Except for changing the geometric parameters to those of a coaxial transmission line, this module offers the same output information as Module 2.1.
Lossless Microstrip Line

Phase velocity in dielectric:

\[ u_p = \frac{c}{\sqrt{\varepsilon_r}} \]

Phase velocity for microstrip:

\[ u_p = \frac{c}{\sqrt{\varepsilon_{\text{eff}}}} \]

\[ \varepsilon_{\text{eff}} = \frac{\varepsilon_r + 1}{2} + \left(\frac{\varepsilon_r - 1}{2}\right) \left(1 + \frac{10}{s}\right)^{-xy}, \quad (2.36) \]

where \( s \) is the width-to-thickness ratio.

\[ s = \frac{w}{h}, \quad (2.37) \]

and \( x \) and \( y \) are intermediate variables given by

\[ x = 0.56 \left[ \frac{\varepsilon_r - 0.9}{\varepsilon_r + 3} \right]^{0.05}, \quad (2.38a) \]

\[ y = 1 + 0.02 \ln \left( \frac{s^4 + 3.7 \times 10^{-4} s^2}{s^4 + 0.43} \right) + 0.05 \ln(1 + 1.7 \times 10^{-4} s^3). \quad (2.38b) \]

The characteristic impedance of the microstrip line is given by
Microstrip (cont.)

The characteristic impedance of the microstrip line is given by

\[
Z_0 = \frac{60}{\sqrt{\varepsilon_{\text{eff}}}} \ln \left\{ \frac{6 + (2\pi - 6)e^{-t}}{s} + \sqrt{1 + \frac{4}{s^2}} \right\}, \quad (2.39)
\]

\[
t = \left( \frac{30.67}{s} \right)^{0.75}
\]

\[
R' = 0 \quad \text{(because } \sigma_c = \infty),
\]

\[
G' = 0 \quad \text{(because } \sigma = 0),
\]

\[
C' = \frac{\sqrt{\varepsilon_{\text{eff}}}}{Z_0 c},
\]

\[
L' = Z_0^2 C',
\]

\[
\alpha = 0 \quad \text{(because } R' = G' = 0),
\]

\[
\beta = \frac{\omega}{c} \sqrt{\varepsilon_{\text{eff}}},
\]
Inverse process: Given $Z_0$, find $s$

The solution formulas are based on two numerical fits, defined in terms of the value of $Z_0$ relative to that of the effective permittivity.

(a) For $Z_0 \leq (44 - 2\varepsilon_r) \Omega$,

$$s = \frac{w}{h} = \frac{2}{\pi} \left\{ (q - 1) - \ln(2q - 1) + \frac{\varepsilon_r - 1}{2\varepsilon_r} \left[ \ln(q - 1) + 0.29 - \frac{0.52}{\varepsilon_r} \right] \right\}$$

(2.42a)

with

$$q = \frac{60\pi^2}{Z_0\sqrt{\varepsilon_r}}$$

(2.42b)

and

(b) for $Z_0 \geq (44 - 2\varepsilon_r) \Omega$,

$$s = \frac{w}{h} = \frac{8e^p}{e^{2p} - 2}$$

(2.43a)

with

$$p = \sqrt{\frac{\varepsilon_r + 1}{2}} \frac{Z_0}{60} + \left( \frac{\varepsilon_r - 1}{\varepsilon_r + 1} \right) \left( 0.23 + \frac{0.12}{\varepsilon_r} \right)$$

(2.43b)
Example 2-2: Microstrip Line

A 50-Ω microstrip line uses a 0.5-mm–thick sapphire substrate with $\varepsilon_r = 9$. What is the width of its copper strip?

**Solution:** Since $Z_0 = 50 > 44 - 18 = 32$, we should use Eq. (2.43):

\[
p = \sqrt{\frac{\varepsilon_r + \frac{1}{2}}{2}} \times \frac{Z_0}{60} + \left( \frac{\varepsilon_r - 1}{\varepsilon_r + 1} \right) \left( 0.23 + \frac{0.12}{\varepsilon_r} \right)
\]

\[
= \sqrt{\frac{9 + \frac{1}{2}}{2}} \times \frac{50}{60} + \left( \frac{9 - 1}{9 + 1} \right) \left( 0.23 + \frac{0.12}{9} \right)
\]

\[
= 2.06,
\]

\[
s = \frac{w}{h}
\]

\[
= \frac{8e^p}{e^{2p} - 2}
\]

\[
= \frac{8e^{2.06}}{e^{4.12} - 2}
\]

\[
= 1.056.
\]

Hence,

\[
w = sh
\]

\[
= 1.056 \times 0.5 \text{ mm}
\]

\[
= 0.53 \text{ mm}.
\]

To check our calculations, we will use $s = 1.056$ to calculate $Z_0$ to verify that the value we obtained is indeed equal or close to 50 Ω. With $\varepsilon_r = 9$, Eqs. (2.36) to (2.40) yield

\[
x = 0.55,
\]

\[
y = 0.99,
\]

\[
t = 12.51,
\]

\[
\varepsilon_{\text{eff}} = 6.11,
\]

\[
Z_0 = 49.93 \Omega.
\]

The calculated value of $Z_0$ is, for all practical purposes, equal to the value specified in the problem statement.
CD Module 2.3 Lossless Microstrip Line The output panel lists the values of the transmission line parameters and displays the variation of $Z_0$ and $\varepsilon_{\text{eff}}$ with $h$ and $w$.

Module 2.3 Lossless Microstrip Line

$$f = 1.794 \text{ [GHz]}$$

Effective Relative Permittivity

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**Input**

- Strip width $w = 1.276$ [mm]
- Substrate thickness $h = 0.635$ [mm]
- Frequency $f = 1.794E9$ [Hz]

**Output**

**Structure Data**

- $h = 0.635$ [mm]
- $w = 1.276$ [mm]
- $w/h = 2.009$

- $Z_0 = 33.324$ [Ω]
- $\varepsilon_{\text{eff}} = 7.074$
- $u_p = 1.128 \times 10^8$ [m/s]
- $\lambda = 0.063$ [m]

- $C' = 266.037$ [pF/m]
- $L' = 295.433$ [nH/m]
- $R' = 0$ [Ω/m]
- $G' = 0$ [S/m]

- $\alpha = 0$ [Np/m]
- $\beta = 99.932$ [rad/m]
Thus, the fact that we can get standing waves on a transmission line (a consequence of losses; dest to interfere) means that in designing a system to transport signals, we need to be careful about the "design" of the transmission line — we will see this later when we discuss explicitly the load.
It is easier to get a better understanding of what this all means if we consider a simple case. Since we can make lines where there is no attenuation of the wave propagating through the line, we say the transmission line is "lossless." If \( \omega \neq 0 \), we say the transmission line is "lossy." Now \( \chi = \sqrt{(\omega' + j\omega L')(\omega' + j\omega C')} = \omega + j\beta \)

After we can approach a "lossless" line, if the conducting material has a high conductivity, \( \varepsilon \rightarrow \infty \), \( \therefore \) the intrinsic resistance \( R_s = \sqrt{\frac{V}{I}} \rightarrow 0 \) thus \( R' \propto R_s \rightarrow 0 \).

and if the insulator between the forward 1 return lines is good, \( \varepsilon \rightarrow 20 \) and \( \mu \propto \sigma \rightarrow 0 \) of insulator.
\[ j = \sqrt{-1} \]

\[ x = \sqrt{jwL'}(\omega_c jw) = \sqrt{j^2w^2L'C'} = jw\sqrt{L'C'} \]

But we can still have a lossless line if \( R' \), \( G' \) are very small but not necessarily 0.

IE: if \( R' < \omega wL' \) and \( G' < \omega wC' \),

then \( R' + jwL' \approx \frac{R'}{\omega wL'} \cdot jwL' \)

\( G' + jwC' \approx \frac{G'}{\omega wC'} \cdot jwC' \)

Thus \( \boxed{x \approx jw\sqrt{L'C'}} \)

In this case also \( \boxed{R' \ll \omega wL'} \)

so it's really \( \frac{R'}{L'} \ll \omega \).

\( G' \ll \omega w \)

which determines if the line will be considered lossless.
the characteristic impedance for this line is

\[ Z_0 = \sqrt{\frac{L'}{G'} + j \omega C'} = \sqrt{\frac{L'}{C'}} = Z_0 \]

this is now real.

\[ u_p = \frac{\lambda_f}{B} = \frac{2\pi f}{B} = \frac{\omega}{c} \]

we can get near the phase velocity of the signal

on a lossless line: \( u_p = \frac{\omega}{c} = \frac{1}{\sqrt{\omega L' C'}} = u_p \)

and \( \lambda = \frac{2\pi f}{u_p} = \frac{2\pi f}{\omega} \sqrt{\frac{C'}{L'}} = \frac{2\pi f}{u_p} \) c. Cut time \( \omega = 2\pi f \)

= \( u_p/f \) ! on standard!
rememering from last lecture that for all TEM trans lines:

\[
\begin{array}{c}
L'c' = MC \\
G'c' = Gc
\end{array}
\]

units are from table 2.1 pp 41

then we can rewrite \( \beta \) and \( u_p \) as

\[
\beta = \omega \sqrt{lc} = \omega \sqrt{MC} \quad \text{in rad/m}
\]

\[
u_p = \frac{1}{\sqrt{lc'}} = \frac{1}{\sqrt{MC}} \quad \text{in m/sec}
\]

\( M, c \) are properties of materials separating the medium.

convince yourself that units are correct.
Lossless Transmission Line

\[\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')}\]

If \(R' \ll \omega L'\) and \(G' \ll \omega C'\)

Then:

\[\gamma = \alpha + j\beta = j\omega \sqrt{L'C'}\], \hspace{1cm} (2.44)

which in turn implies that

\[
\begin{align*}
\alpha &= 0 \quad \text{(lossless line),} \\
\beta &= \omega \sqrt{L'C'} \quad \text{(lossless line).} \\
\end{align*}
\] \hspace{1cm} (2.45)

For the characteristic impedance, application of the lossless line conditions to Eq. (2.29) leads to

\[Z_0 = \sqrt{\frac{L'}{C'}} \quad \text{(lossless line),} \hspace{1cm} (2.46)\]

\[\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega \sqrt{L'C'}}\]

\[u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}}\]

\[\beta = \omega \sqrt{\mu \varepsilon} \quad \text{(rad/m),} \hspace{1cm} (2.49)\]

\[u_p = \frac{1}{\sqrt{\mu \varepsilon}} \quad \text{(m/s),} \hspace{1cm} (2.50)\]

If sinusoidal waves of different frequencies travel on a transmission line with the same phase velocity, the line is called **nondispersive**.

\[\lambda = \frac{u_p}{f} = \frac{c}{f} \frac{1}{\sqrt{\varepsilon_r}} = \frac{\lambda_0}{\sqrt{\varepsilon_r}}\]
### Table 2-2: Characteristic parameters of transmission lines.

<table>
<thead>
<tr>
<th></th>
<th>Propagation Constant $\gamma = \alpha + j\beta$</th>
<th>Phase Velocity $u_p = \omega / \beta$</th>
<th>Characteristic Impedance $Z_0 = \frac{(R' + j\omega L')}{(G' + j\omega C')}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>General case</strong></td>
<td>$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')}$</td>
<td>$u_p = \omega / \beta$</td>
<td>$Z_0 = \frac{(R' + j\omega L')}{(G' + j\omega C')}$</td>
</tr>
<tr>
<td><strong>Lossless</strong> $(R' = G' = 0)$</td>
<td>$\alpha = 0, \beta = \omega \sqrt{\varepsilon_r} / c$</td>
<td>$u_p = c / \sqrt{\varepsilon_r}$</td>
<td>$Z_0 = \sqrt{L' / C'}$</td>
</tr>
<tr>
<td><strong>Lossless coaxial</strong></td>
<td>$\alpha = 0, \beta = \omega \sqrt{\varepsilon_r} / c$</td>
<td>$u_p = c / \sqrt{\varepsilon_r}$</td>
<td>$Z_0 = \left(\frac{60}{\sqrt{\varepsilon_r}}\right) \ln(b/a)$</td>
</tr>
<tr>
<td><strong>Lossless two-wire</strong></td>
<td>$\alpha = 0, \beta = \omega \sqrt{\varepsilon_r} / c$</td>
<td>$u_p = c / \sqrt{\varepsilon_r}$</td>
<td>$Z_0 = \left(\frac{120}{\sqrt{\varepsilon_r}}\right) \ln\left(\frac{b}{a}\right) + \sqrt{\left(\frac{b}{a}\right)^2 - 1}$ if $D \gg d$</td>
</tr>
<tr>
<td><strong>Lossless parallel-plate</strong></td>
<td>$\alpha = 0, \beta = \omega \sqrt{\varepsilon_r} / c$</td>
<td>$u_p = c / \sqrt{\varepsilon_r}$</td>
<td>$Z_0 = \left(\frac{120\pi}{\sqrt{\varepsilon_r}}\right) \left(\frac{h}{w}\right)$</td>
</tr>
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</table>

**Notes:**
1. $\mu = \mu_0$, $\varepsilon = \varepsilon_r\varepsilon_0$, $c = 1 / \sqrt{\mu_0\varepsilon_0}$, and $\sqrt{\mu_0 / \varepsilon_0} \simeq (120\pi)$ $\Omega$, where $\varepsilon_r$ is the relative permittivity of insulating material.
2. For coaxial line, $a$ and $b$ are radii of inner and outer conductors.
3. For two-wire line, $d = $ wire diameter and $D = $ separation between wire centers.
4. For parallel-plate line, $w = $ width of plate and $h = $ separation between the plates.
and non-dissipative means \( \mathbf{v} \mathbf{u} \rightarrow \mathbf{v} \mathbf{u} \) over line

Now let us look at the uniqueness of transmission line characteristics in the transfer of signal/power from a "source" to a "load" —

**NOTE** we will use nomenclature from text where
- the load is at position \( z = 0 \) and
- source is at position \( z = -L \)

(some texts may use opposite nomenclature — result will be the same — but easier to follow test if we do it same way)

Consider the general case — we will go directly to the phasor representation
Note that because of the transmission line, $\vec{I}_L \neq \vec{I}_i$ in general (remember waves (signal) going in both directions).

We want to evaluate the total voltage in the line at any point $z$. 


**Table 2-2:** Characteristic parameters of transmission lines.

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<td><strong>Lossless</strong></td>
<td>$\alpha = 0, \beta = \omega \sqrt{\varepsilon_r} / c$</td>
<td>$u_p = c / \sqrt{\varepsilon_r}$</td>
<td>$Z_0 = \sqrt{L' / C'}$</td>
</tr>
<tr>
<td>$(R' = G' = 0)$</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td><strong>Lossless coaxial</strong></td>
<td>$\alpha = 0, \beta = \omega \sqrt{\varepsilon_r} / c$</td>
<td>$u_p = c / \sqrt{\varepsilon_r}$</td>
<td>$Z_0 = (60 / \sqrt{\varepsilon_r}) \ln (b / a)$</td>
</tr>
<tr>
<td><strong>Lossless two-wire</strong></td>
<td>$\alpha = 0, \beta = \omega \sqrt{\varepsilon_r} / c$</td>
<td>$u_p = c / \sqrt{\varepsilon_r}$</td>
<td>$Z_0 = \left(120 / \sqrt{\varepsilon_r}\right) \cdot \ln \left(\frac{D}{d}\right) + \sqrt{\left(\frac{D}{d}\right)^2 - 1}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$Z_0 \simeq \left(120 / \sqrt{\varepsilon_r}\right) \ln \left(\frac{2D}{d}\right)$, if $D \gg d$</td>
</tr>
<tr>
<td><strong>Lossless parallel-plate</strong></td>
<td>$\alpha = 0, \beta = \omega \sqrt{\varepsilon_r} / c$</td>
<td>$u_p = c / \sqrt{\varepsilon_r}$</td>
<td>$Z_0 = \left(120\pi / \sqrt{\varepsilon_r}\right) (h / w)$</td>
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**Notes:**
1. $\mu = \mu_0$, $\varepsilon = \varepsilon_r \varepsilon_0$, $c = 1 / \sqrt{\mu_0 \varepsilon_0}$, and $\sqrt{\mu_0 / \varepsilon_0} \simeq (120\pi) \, \Omega$, where $\varepsilon_r$ is the relative permittivity of insulating material.
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4. For parallel-plate line, $w =$ width of plate and $h =$ separation between the plates.
\[
\tilde{V}(z) = V_0^+ e^{-\frac{z}{Z_0}} + V_0^- e^{\frac{z}{Z_0}} \quad \text{remember } V_0^\pm \text{ can be complex}
\]

\[
\tilde{I}(z) = I_0^+ e^{-\frac{z}{Z_0}} + I_0^- e^{\frac{z}{Z_0}}
\]

but from before \( I_0^+ = \frac{V_0^+}{Z_0} \quad I_0^- = -\frac{V_0^-}{Z_0} \) where \( Z_0 = \text{chaos impedance of the line} \)

\[
\tilde{V}(z) = V_0^+ e^{-\frac{z}{Z_0}} + V_0^- e^{\frac{z}{Z_0}} \quad \text{distance from load}
\]

\[
\tilde{I}(z) = \frac{1}{Z_0} [V_0^+ e^{-\frac{z}{Z_0}} - V_0^- e^{\frac{z}{Z_0}}]
\]

at the load, the impedance is \( Z_L = Z(z=0) = \frac{\tilde{V}_L}{\tilde{I}_L} \)

\[
Z_L = \frac{\tilde{V}_L}{\tilde{I}_L} = \frac{V_0^+ + V_0^-}{I_0^+ - I_0^-} = \frac{Z_0}{V_0^+ - V_0^-} = \frac{Z_0}{V_0^+ - V_0^-}
\]

\[
\text{can be complex even tho } Z_0 \text{ is real since } V_0^+ \text{ may be complex}
\]

rewriting \( Z_L \) gives us:

\[
\text{Forward wave} \quad \frac{V_0^-}{V_0^+} = \left( \frac{Z_L - Z_0}{Z_L + Z_0} \right) = \frac{\frac{Z_L}{Z_0} - 1}{\frac{Z_L}{Z_0} + 1}
\]

\[
\text{Reflective wave} \quad \text{the voltage ratio of the backward "wave" to the forward "wave"}
\]

\[
\text{note: this reflective wave only depends on the ratio of load impedance to chaos impedance of line}
\]

\[
a \text{load is "matched" if } \Gamma = 0 \text{ then no reflected "wave" (signal) - ie, all current, voltage delived to load!}
\]
we can now do a similar analysis for the current $I$
noting that since
\[
\frac{V_0^+}{I_0^+} = Z_0 = -\frac{V_0^-}{I_0^-}
\]
then
\[
\begin{align*}
I_0^- &= \frac{V_0^-}{V_0^+} = -I_0^+ \\
I_0^+ &= \frac{V_0^+}{V_0^-} = -I_0^-
\end{align*}
\]
so the condition for "reflection" at the load
(i.e., that we get a right-to-left wave) depends
upon the nature of the load ($Z_L$) and $Z_L$ can be real (resistive),
reactive (imaginary), or complex (reactive).
-or complex
Since $\Gamma$ can be complex, we write $\Gamma$ in polar form
\[
\Gamma = |\Gamma| e^{i\Theta}
\]
where $\Theta$ is the phase angle associated with the signal at the load.

Look at the different cases:
1. $Z_L = Z_0$ a "matched" load. $\Gamma = 0$ no reflection. $V_0^- = I_0^- = 0$
2. $Z_L = 0$ an "open circuit". $\Gamma = 1$ all $V_0^- = V_0^+$ (in phase)
3. $Z_L = \infty$ a "short circuit". $\Gamma = -1$ inverted in phase. $V_0^- = -V_0^+$

Why do we care?
Power = IV
So to transfer more power
Would consider using a "buffer" matching the two impedances

\text{unit}
Voltage Reflection Coefficient

\[ \begin{align*}
\tilde{V}_L &= \tilde{V}(z=0) = V_0^+ + V_0^- , \\
\tilde{I}_L &= \tilde{I}(z=0) = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0} .
\end{align*} \]

At the load \((z = 0)\):

\[ Z_L = \frac{\tilde{V}_L}{\tilde{I}_L} \]

\[ \begin{align*}
\tilde{V}_L &= \tilde{V}(z=0) = V_0^+ + V_0^- , \\
\tilde{I}_L &= \tilde{I}(z=0) = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0} .
\end{align*} \]

Using these expressions in Eq. (2.55), we obtain

\[ Z_L = \left( \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} \right) Z_0 . \]

\[ \Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} \]

\[ = \frac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1} \]

\[ = \frac{z_L - 1}{z_L + 1} \quad \text{(dimensionless),} \quad (2.59) \]

Normalized load impedance
## Voltage Reflection Coefficient

\[ \Gamma = |\Gamma| e^{j\theta_r} \]

### Reflection Coefficient

| Load          | \(|\Gamma|\) | \(\theta_r\)                          |
|---------------|-------------|---------------------------------------|
| \(Z_0\) \(Z_L = (r + jx)Z_0\) | \(\left[\frac{(r - 1)^2 + x^2}{(r + 1)^2 + x^2}\right]^{1/2}\) | \(\tan^{-1}\left(\frac{x}{r - 1}\right) - \tan^{-1}\left(\frac{x}{r + 1}\right)\) |
| \(Z_0\) \(Z_0\)          | 0 (no reflection) | irrelevant                             |
| \(Z_0\) (short)           | 1             | \(\pm 180^\circ\) (phase opposition)  |
| \(Z_0\) (open)            | 1             | 0 (in-phase)                           |
| \(Z_0\) \(jX = j\omega L\) | 1             | \(\pm 180^\circ - 2\tan^{-1} x\)      |
| \(Z_0\) \(jX = \frac{-j}{\omega C}\) | 1             | \(\pm 180^\circ + 2\tan^{-1} x\)      |

\[ z_L = \frac{Z_L}{Z_0} = (R + jX)/Z_0 = r + jx \]
case 4. pure "reactive" load

\[ Z_L = jX_L \]

Then \[ \Gamma = \frac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1} = \frac{X_L/Z_0 - 1}{X_L/Z_0 + 1} = \frac{X_L - Z_0}{X_L + Z_0} \]

\[ \Gamma = -\frac{Z_0 - jX_L}{Z_0 + jX_L} = -\frac{\sqrt{Z_0^2 + X_L^2} e^{-j\theta}}{\sqrt{Z_0^2 + X_L^2} e^{j\theta}} \]

\[ Z_0 + jX_L = \sqrt{Z_0^2 + X_L^2} e^{j\theta} \]

writing in polar form

\[ \Gamma = -e^{-2j\theta} \]

\[ |\Gamma| = 1 \]

\[ \theta = \text{atan2}(X_L, Z_0) \]

for a mixed load \[ Z_L = R_L + jX_L \]

we will get \(|\Gamma| < 1\) with a phase shift.

ie partial reflection, change of phase

note that \(|\Gamma|^2\) will tell us the fraction of power delivered to the load. since \( P = EV \) hence on this

the fact that we can send a signal down a "transmit line" and get "reflected" signals back in the line means we can get interference and hence, "standing waves".

remember: a standing wave is one where the nodes are fixed in space, indep. of time.
Current Reflection Coefficient

\[
\frac{I_0^-}{I_0^+} = -\frac{V_0^-}{V_0^+} = -\Gamma. \quad (2.61)
\]

We note that whereas the ratio of the voltage amplitudes is equal to \( \Gamma \), the ratio of the current amplitudes is equal to \(-\Gamma\).
Example 2-3: Reflection Coefficient of a Series RC Load

A 100-Ω transmission line is connected to a load consisting of a 50-Ω resistor in series with a 10-pF capacitor. Find the reflection coefficient at the load for a 100-MHz signal.

**Solution:** The following quantities are given (Fig. 2-13):

\[ R_L = 50 \, \Omega, \quad C_L = 10 \, \text{pF} = 10^{-11} \, \text{F}, \]

\[ Z_0 = 100 \, \Omega, \quad f = 100 \, \text{MHz} = 10^8 \, \text{Hz}. \]

The normalized load impedance is

\[ z_L = \frac{Z_L}{Z_0} = \frac{R_L - j/(\omega C_L)}{Z_0} = \frac{1}{100} \left( 50 - j \frac{1}{2\pi \times 10^8 \times 10^{-11}} \right) = (0.5 - j1.59) \, \Omega. \]

\[ \Gamma = \frac{z_L - 1}{z_L + 1} = \frac{0.5 - j1.59 - 1}{0.5 - j1.59 + 1} = \frac{-0.5 - j1.59}{1.5 - j1.59} \]

\[ = \frac{-0.5 - j1.59}{1.5 - j1.59} \frac{1.5 + j1.59}{1.5 + j1.59} = \frac{-1.67e^{j72.6^\circ}}{2.19e^{-j46.7^\circ}} = -0.76e^{j119.3^\circ}. \]

This result may be converted into the form of Eq. (2.62) by replacing the minus sign with \( e^{-j180^\circ} \). Thus,

\[ \Gamma = 0.76e^{j119.3^\circ} e^{-j180^\circ} = 0.76e^{-j60.7^\circ} = 0.76 \angle -60.7^\circ, \]

or

\[ |\Gamma| = 0.76, \quad \theta_\Gamma = -60.7^\circ. \]
so let us look at the magnitude of the total signal
voltage propagating on the line (in phase
frq domain)

\[ |\tilde{V}(t)| = \sqrt{\tilde{V}(t) \cdot \tilde{V}(t)^*} \]

\[ \text{complex conjugate of a complex } \]

for simplicity we will assume a "lossless" line: i.e. \( v=0, x=0 \beta \)

\[ \tilde{V}(t) = v_0 e^{-\beta z} + v_0' e^{\beta z} \]

\[ = v_0' \left[ e^{-\beta z} + v_0 e^{\beta z} \right] = v_0' \left[ e^{-\beta z} + (1) e^{\beta z} \right] \]

\[ \tilde{V}(t) = v_0' \left[ e^{-\beta z} + \left[ 1 \Re \{ e^{\beta z} e^{j\theta} \} \right] \right] \]

\[ = \left\{ v_0' \left[ e^{-\beta z} + \left[ 1 \Re \{ e^{\beta z} e^{j\theta} \} \right] \right] \right\}^{1/2} \]

\[ |\tilde{V}(t)| = \sqrt{\left\{ v_0' \left[ e^{-\beta z} + \left[ 1 \Re \{ e^{\beta z} e^{j\theta} \} \right] \right] \right\}^{1/2}} \]

\[ = \sqrt{v_0'^2 \left[ 1 + (1) \Re \{ e^{\beta z} e^{j\theta} \} \right]} \]

\[ \text{remember that: } e^{JA} + e^{-JA} = 2 J A \text{ then with } A = \theta_k \pm 2 \beta z \text{ we get} \]

\[ |\tilde{V}(t)| = \left| v_0' \left[ 1 + (1) \Re \{ e^{\beta z} e^{j\theta} \} \right] \right|^{1/2} \]

\[ \tilde{V}(t) = v_0' e^{\beta z} \text{ when we go back to the time domain, we will then get} \]

\[ \text{this must be time varying amp. thus modes will} \]

\[ \text{be time independent and will occur when } \beta z \pm \theta_k = -(2n+1)\pi, n=0,1,2,.. \]

\[ \text{ie we get } \beta z \text{ negative } \]
Standing Waves

Using the relation $V_0^- = \Gamma V_0^+$ yields

$$\tilde{V}(z) = V_0^+ (e^{-j\beta z} + \Gamma e^{j\beta z}),$$

$$\tilde{I}(z) = \frac{V_0^+}{Z_0} (e^{-j\beta z} - \Gamma e^{j\beta z}).$$

$$|\tilde{V}(z)| = \left\{ \left[ V_0^+ (e^{-j\beta z} + |\Gamma| e^{j\theta_e} e^{j\beta z}) \right] \cdot \left[ (V_0^+)^* (e^{j\beta z} + |\Gamma| e^{-j\theta_e} e^{-j\beta z}) \right] \right\}^{1/2}
= |V_0^+| \left[ 1 + |\Gamma|^2 + |\Gamma|(e^{j(2\beta z + \theta_e)} + e^{-j(2\beta z + \theta_e)}) \right]^{1/2}
= |V_0^+| \left[ 1 + |\Gamma|^2 + 2|\Gamma| \cos(2\beta z + \theta_e) \right]^{1/2}, \quad (2.64)$$

To express the magnitude of $\tilde{V}$ as a function of $d$ instead of $z$, we replace $z$ with $-d$ on the right-hand side of Eq. (2.64):

$$|\tilde{V}(d)| = |V_0^+| \left[ 1 + |\Gamma|^2 + 2|\Gamma| \cos(2\beta d - \theta_e) \right]^{1/2}. \quad (2.66)$$

**Voltage magnitude**

By applying the same steps to Eq. (2.63b), a similar expression can be derived for $|\tilde{I}(d)|$, the magnitude of the current $\tilde{I}(d)$:

$$|\tilde{I}(d)| = \frac{|V_0^+|}{Z_0} \left[ 1 + |\Gamma|^2 - 2|\Gamma| \cos(2\beta d - \theta_e) \right]^{1/2}. \quad (2.67)$$

**Current magnitude**
Standing-Wave Pattern

Whereas the repetition period is $\lambda$ for the incident and reflected waves considered individually, the repetition period of the standing-wave pattern is $\lambda/2$.

\[ |\tilde{V}(d)| = |V_0^+| \left[ 1 + |\Gamma|^2 + 2|\Gamma| \cos(2\beta d - \theta_r) \right]^{1/2}. \quad (2.66) \]

\[ |\tilde{I}(d)| = \frac{|V_0^+|}{Z_0} \left[ 1 + |\Gamma|^2 - 2|\Gamma| \cos(2\beta d - \theta_r) \right]^{1/2}. \quad (2.67) \]

Voltage magnitude is maximum when $(2\beta d_{\text{min}} - \theta_r) = (2n + 1)\pi$.

When voltage is a maximum, current is a minimum, and vice versa.

Figure 2-14: Standing-wave pattern for (a) $|\tilde{V}(d)|$ and (b) $|\tilde{I}(d)|$ for a lossless transmission line of characteristic impedance $Z_0 = 50 \, \Omega$, terminated in a load with a reflection coefficient $\Gamma = 0.3e^{j30^\circ}$. The magnitude of the incident wave $|V_0^+| = 1 \, V$. The standing-wave ratio is $S = |\tilde{V}|_{\text{max}}/|\tilde{V}|_{\text{min}} = 1.3/0.7 = 1.86.$
1. In these voltage patterns for specific cases:
   (Remember, we are still dealing with a lossless line)

1) A "matched line". $Z_L = Z_0 \Rightarrow \Gamma = 0$
   
   $|V(z)| = \left| V_0 \right|^2 \left( \frac{1}{3} \right) \frac{1}{2} = \left| V_0 \right|$  no variation in z

   ![Diagram showing voltage at output load with \( z = 0 \) and \( \Gamma = 0 \)]

2) A "short circuit". $Z_L = 0 \Rightarrow \Gamma = -1$
   
   Since $\Gamma = \left( \frac{\omega L}{\omega C} \right)$, this means $\theta_L = \pi, 3\pi, 5\pi$
   
   The reflected "signal" is $\pi$ out of phase.

   $|V(z)| = \left| V_0 \right|^2 \left( 1 + Z_0 \left( 2\beta \right) + \theta_L \right) \frac{1}{2}$

   $\therefore$ minima occur when $2\beta + \theta_L = \pm (2\pi n + 1)$  $n = 0, 1, 2, \ldots$
   
   Since $\beta = \frac{2\pi}{\lambda}$, wavelength of the signal

   $2\beta = \frac{4\pi}{\lambda}$

   $\therefore 2\beta = \pm (2\pi n + 1) - \theta_L$

   $z = \left[ \pm (2\pi n + 1) - \theta_L \right] \frac{\lambda}{4\pi}$  we only care about $-z$ values
   
   \[ \text{(Case of load)} \]

   $-z = \left[ \pm (2\pi n + 1) + \theta_L \right] \frac{\lambda}{4\pi} = \frac{1}{4\pi} \left[ \pm (2\pi n + 1) \lambda + \theta_L \frac{\lambda}{4\pi} \right]$

   $\therefore -z = (2\pi n + 1) \lambda + \theta_L \frac{\lambda}{4\pi}$

   $\therefore -z = \left( \frac{\lambda}{4\pi} \right) \left( \frac{n + \frac{1}{4}}{\pi} \right) \lambda = \lambda^2 + \frac{n\lambda}{z} \lambda = \lambda^2 + \frac{n\lambda}{z}$

   \[ \text{min} \]
so for short circuit min occur every \( \frac{n \lambda}{2} \)

\[ -\frac{3 \lambda}{4}, -\frac{\lambda}{4}, -\frac{5 \lambda}{4}, -\frac{3 \lambda}{4}, \frac{\lambda}{4}, \frac{5 \lambda}{4}, \frac{3 \lambda}{4}, 0 \]

and you can show for maxima:

\[-Z = \frac{\lambda}{4} + \frac{n \lambda}{2}, \quad n = 0, 1, 2, 3\]

we can then do a similar analysis for an open circuit load

\[ Z_L = \infty \quad \text{and} \quad \Gamma = 1 \quad \text{(in phase reflection)}\]

\[ \text{where} \quad \theta_R = 0 \]

we then get max at \(-Z = \frac{n \lambda}{2}, n = 0, 1, 2\)

\(-Z = \frac{\lambda}{4} + \frac{n \lambda}{2} \quad \text{(just opposite short circuit case)}\)

matched \(\rightarrow\)

\[ |V(z)| \]

open \(\rightarrow\)

\[ |V(z)| \]

one defines a quantity \(S = \text{VSWR (vswr)} = \frac{|V_{\text{max}}|}{|V_{\text{min}}|}\)

the voltage standing wave ratio as a measure of the mismatch of the load to line

\[ S = \frac{1+|\Gamma|}{1-|\Gamma|} \]

\(S=1\) for matched line (\(\Gamma=0\))

\(S=\infty\) for complete mismatch (\(|\Gamma|=1\)) open or short
Standing Wave Patterns for 3 Types of Loads

With no reflected wave present, there will be no interference and no standing waves.

Example 2-4: $|\Gamma|$ for Purely Reactive Load

Show that $|\Gamma| = 1$ for a lossless line connected to a purely reactive load.

Solution: The load impedance of a purely reactive load is $Z_L = jX_L$.

From Eq. (2.59), the reflection coefficient is

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{jX_L - Z_0}{jX_L + Z_0}$$

$$= \frac{- (Z_0 - jX_L)}{(Z_0 + jX_L)} = \frac{-\sqrt{Z_0^2 + X_L^2} e^{-j\theta}}{\sqrt{Z_0^2 + X_L^2} e^{j\theta}} = -e^{-j2\theta},$$

where $\theta = \tan^{-1} X_L/Z_0$. Hence

$$|\Gamma| = |-e^{-j2\theta}| = [(e^{-j2\theta})(e^{-j2\theta})^*]^{1/2} = 1.$$
Maxima & Minima

Standing-Wave Pattern

Let us denote $d_{\text{max}}$ as the distance from the load at which $|\tilde{V}(d)|$ is a maximum. It then follows that

$$|\tilde{V}(d)| = |\tilde{V}_{\text{max}}| = |V_0^+| |1 + |\Gamma||,$$  \hspace{0.5cm}  (2.68)

when

$$2\beta d_{\text{max}} - \theta_r = 2n\pi,$$  \hspace{0.5cm}  (2.69)

with $n = 0$ or a positive integer. Solving Eq. (2.69) for $d_{\text{max}}$, we have

$$d_{\text{max}} = \frac{\theta_r + 2n\pi}{2\beta} = \frac{\theta_r \lambda}{4\pi} + \frac{n\lambda}{2},$$  \hspace{0.5cm}  \begin{cases} n = 1, 2, \ldots & \text{if } \theta_r < 0, \\ n = 0, 1, 2, \ldots & \text{if } \theta_r \geq 0, \end{cases}  \hspace{0.5cm} (2.70)
Maxima & Minima (cont.)

\[ |\tilde{V}|_{\text{min}} = |V_0^+| [1 - |\Gamma|], \]
when \(2 \beta d_{\text{min}} - \theta_r = (2n + 1)\pi\)

\[
S = \frac{|\tilde{V}|_{\text{max}}}{|\tilde{V}|_{\text{min}}} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \quad \text{(dimensionless)}
\]

\[ S = \text{Voltage Standing Wave Ratio} \]

For a matched load: \[ S = 1 \]

For a short, open, or purely reactive load:

\[ S = \infty \]
CD Module 2.4 Transmission Line Simulator Upon specifying the requisite input data—including the load impedance at $d = 0$ and the generator voltage and impedance at $d = l$, this module provides a wealth of output information about the voltage and current waveforms along the transmission line. You can view plots of the standing wave patterns for voltage and current, the time and spatial variations of the instantaneous voltage $v(d, t)$ and current $i(d, t)$, and other related quantities.
Example 2-6: Measuring $Z_L$ with a Slotted Line

Next, we use the condition given by Eq. (2.71) to find $\theta_r$:

$$2\beta d_{\text{min}} - \theta_r = \pi,$$

for $n = 0$ (first minimum),

which gives

$$\theta_r = 2\beta d_{\text{min}} - \pi$$

$$= 2 \times \frac{10\pi}{3} \times 0.12 - \pi$$

$$= -0.2\pi \text{ (rad)}$$

$$= -36^\circ.$$  

Hence,

$$\Gamma = |\Gamma|e^{j\theta_r}$$

$$= 0.5e^{-j36^\circ}$$

$$= 0.405 - j0.294.$$  

Solving Eq. (2.59) for $Z_L$, we have

$$Z_L = Z_0 \left[ \frac{1 + \Gamma}{1 - \Gamma} \right]$$

$$= 50 \left[ \frac{1 + 0.405 - j0.294}{1 - 0.405 + j0.294} \right]$$

$$= (85 - j67) \Omega.$$  

Solution: The following quantities are given:

$$Z_0 = 50 \ \Omega,$$

$$S = 3,$$

$$d_{\text{min}} = 12 \text{ cm}.$$  

Since the distance between successive voltage minima is $\lambda/2$,

$$\lambda = 2 \times 0.3 = 0.6 \text{ m},$$

and

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{0.6} = \frac{10\pi}{3} \text{ (rad/m)}.$$  

From Eq. (2.73), solving for $|\Gamma|$ in terms of $S$ gives

$$|\Gamma| = \frac{S - 1}{S + 1}$$

$$= \frac{3 - 1}{3 + 1}$$

$$= 0.5.$$