EE 135, Winter 2012

Reading: Chapter 5, Magnetostatics, Sections 1-8.

Homework #5, chapter 4; probs. 4.36, 4.44, 4.52, 4.61, Chapter 5; probs. 5.4, 5.8, 5.12, 5.21 due Thursday, Feb. 23, 2012.

Discussion Session: 7-8pm Wednesdays, Jack’s Lounge

Lecture 11
Maxwell’s Equations

\[ \nabla \cdot \mathbf{D} = \rho_v, \]
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \]
\[ \nabla \cdot \mathbf{B} = 0, \]
\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \]

Under static conditions, none of the quantities appearing in Maxwell’s equations are functions of time (i.e., \( \partial / \partial t = 0 \)). This happens when all charges are permanently fixed in space, or, if they move, they do so at a steady rate so that \( \rho_v \) and \( \mathbf{J} \) are constant in time. Under these circumstances, the time derivatives of \( \mathbf{B} \) and \( \mathbf{D} \) in Eqs. (4.1b) and (4.1d) vanish, and Maxwell’s equations reduce to

**Electrostatics**

\[ \nabla \cdot \mathbf{D} = \rho_v, \quad (4.2a) \]
\[ \nabla \times \mathbf{E} = 0. \quad (4.2b) \]

**Magnetostatics**

\[ \nabla \cdot \mathbf{B} = 0, \quad (4.3a) \]
\[ \nabla \times \mathbf{H} = \mathbf{J}. \quad (4.3b) \]

Electric and magnetic fields become decoupled under static conditions.
André-Marie Ampère (1775-1836)

Born
20 January 1775
Parish of St. Nizier, Lyon, France

Died
10 June 1836 (aged 61)
Marseille, France

Residence
France

Nationality
French

Fields
Physics

Institutions
Bourg-en-Bresse
École Polytechnique

Known for
Ampère's Law

Signature
A. Ampère

Hans Christian Ørsted

Born
14 August 1777
Rudkøbing, Denmark

Died
9 March 1851 (aged 73)
Copenhagen, Denmark

Nationality
Danish

Fields
physics, chemistry

Known for
electromagnetism

Signature
H.C. Ørsted
Magnetic Torque on Current Loop

\[ \mathbf{F}_1 = I (\hat{y}b) \times (\hat{x}B_0) = \hat{z}IbB_0, \]

\[ \mathbf{F}_3 = I (\hat{y}b) \times (\hat{x}B_0) = -\hat{z}IbB_0. \]

No forces on arms 2 and 4 (because I and B are parallel, or anti-parallel)

**Magnetic torque:**

\[
T = d_1 \times F_1 + d_3 \times F_3 \\
= \left( -\hat{x} \frac{a}{2} \right) \times (\hat{z}IbB_0) + \left( \hat{x} \frac{a}{2} \right) \times (-\hat{z}IbB_0) \\
= \hat{y}IabB_0 = \hat{y}IAB_0,
\]

Area of Loop

Figure 5-6: Rectangular loop pivoted along the y-axis: (a) front view and (b) bottom view. The combination of forces \( F_1 \) and \( F_3 \) on the loop generates a torque that tends to rotate the loop in a clockwise direction as shown in (b).
Inclined Loop

For a loop with \(N\) turns and whose surface normal is at angle \(\theta\) relative to \(B\) direction:

\[
T = NIA B_0 \sin \theta. \quad (5.18)
\]

The quantity \(NIA\) is called the magnetic moment \(m\) of the loop. Now, consider the vector

\[
m = \hat{n} NIA = \hat{n} m \quad (\text{A} \cdot \text{m}^2), \quad (5.19)
\]

where \(\hat{n}\) is the surface normal of the loop and governed by the following right-hand rule: when the four fingers of the right hand advance in the direction of the current \(I\), the direction of the thumb specifies the direction of \(\hat{n}\). In terms of \(m\), the torque vector \(T\) can be written as

\[
T = m \times B \quad (\text{N} \cdot \text{m}). \quad (5.20)
\]

Figure 5-7: Rectangular loop in a uniform magnetic field with flux density \(B\) whose direction is perpendicular to the rotation axis of the loop, but makes an angle \(\theta\) with the loop’s surface normal \(\hat{n}\).
Jean-Baptiste Biot

1820/ electric current
deflected compass needles
Biot-Savart Law

Magnetic field induced by a differential current:

\[ d\mathbf{H} = \frac{I}{4\pi} \frac{d\mathbf{l} \times \mathbf{\hat{R}}}{R^2} \quad \text{(A/m)} \]

For the entire length:

\[ \mathbf{H} = \frac{I}{4\pi} \int_{\mathcal{L}} \frac{d\mathbf{l} \times \mathbf{\hat{R}}}{R^2} \quad \text{(A/m)}, \quad (5.22) \]

where \( \mathcal{L} \) is the line path along which \( I \) exists.

**Figure 5-8:** Magnetic field \( d\mathbf{H} \) generated by a current element \( I \, d\mathbf{l} \). The direction of the field induced at point \( P \) is opposite to that induced at point \( P' \).
Magnetic Field due to Current Densities

\[ H = \frac{1}{4\pi} \int_S \frac{J_s \times \hat{R}}{R^2} \, ds \]  
(surface current),

\[ H = \frac{1}{4\pi} \int_V \frac{J \times \hat{R}}{R^2} \, dV \]  
(volume current).

**Figure 5-9:** (a) The total current crossing the cross section \( S \) of the cylinder is \( I = \int_S J \cdot ds \). (b) The total current flowing across the surface of the conductor is \( I = \int_I J_s \, dl \).
**Example 5-2: Magnetic Field of Linear Conductor**

**Solution:** From Fig. 5-10, the differential length vector $dl = \hat{z} \, dz$. Hence, $dl \times \hat{R} = dz (\hat{z} \times \hat{R}) = \hat{\phi} \sin \theta \, dz$, where $\hat{\phi}$ is the azimuth direction and $\theta$ is the angle between $dl$ and $\hat{R}$. Application of Eq. (5.22) gives

$$
H = \frac{I}{4\pi} \int_{z=-l/2}^{z=l/2} \frac{dl \times \hat{R}}{R^2} = \hat{\phi} \frac{I}{4\pi} \int_{-l/2}^{l/2} \frac{\sin \theta}{R^2} \, dz. \tag{5.25}
$$

Both $R$ and $\theta$ are dependent on the integration variable $z$, but the radial distance $r$ is not. For convenience, we will convert the integration variable from $z$ to $\theta$ by using the transformations

$$
R = r \csc \theta, \tag{5.26a}
$$

$$
z = -r \cot \theta, \tag{5.26b}
$$

$$
dz = r \csc^2 \theta \, d\theta. \tag{5.26c}
$$

Figure 5-10: Linear conductor of length $l$ carrying a current $I$. (a) The field $dH$ at point $P$ due to incremental current element $dl$. (b) Limiting angles $\theta_1$ and $\theta_2$, each measured between vector $I \, dl$ and the vector connecting the end of the conductor associated with that angle to point $P$ (Example 5-2).
Upon inserting Eqs. (5.26a) and (5.26c) into Eq. (5.25), we have

\[
H = \hat{\phi} \frac{I}{4\pi} \int_{\theta_1}^{\theta_2} \frac{\sin \theta \ csc^2 \theta \ d\theta}{r^2 \ csc^2 \theta}
\]

\[
= \hat{\phi} \frac{I}{4\pi r} \int_{\theta_1}^{\theta_2} \sin \theta \ d\theta
\]

\[
= \hat{\phi} \frac{I}{4\pi r} \left( \cos \theta_1 - \cos \theta_2 \right), \quad (5.27)
\]

where \(\theta_1\) and \(\theta_2\) are the limiting angles at \(z = -l/2\) and \(z = l/2\), respectively. From the right triangle in Fig. 5-10(b), it follows that

\[
\cos \theta_1 = \frac{l/2}{\sqrt{r^2 + (l/2)^2}}, \quad (5.28a)
\]

\[
\cos \theta_2 = -\cos \theta_1 = \frac{-l/2}{\sqrt{r^2 + (l/2)^2}}. \quad (5.28b)
\]

Hence,

\[
B = \mu_0 H = \hat{\phi} \frac{\mu_0 I I}{2\pi r \sqrt{4r^2 + l^2}} \quad (T). \quad (5.29)
\]

For an infinitely long wire with \(l \gg r\), Eq. (5.29) reduces to

\[
B = \hat{\phi} \frac{\mu_0 I}{2\pi r} \quad \text{(infinitely long wire).} \quad (5.30)
\]
Biot-Savart Law

Currents induce magnetic fields.

Observation that current-carrying wires caused compass needles to deflect — ~1820

Remember \( \overrightarrow{B} = \mu \overrightarrow{H} \)

\[
d\mathbf{H} = \frac{I}{4\pi} \frac{d\mathbf{x} \times \hat{R}}{R^2} \quad \text{in amp/m}
\]

\( R = R^2 \) is the distance vector between \( d\mathbf{x} \) and observation point

(Where we want to determine mag. field)

\[
d\mathbf{H} \times \hat{R_1} \text{ is out of page here}
\]

\[
\frac{d\mathbf{H}}{d\mathbf{R_1}} = \frac{dI}{d\mathbf{R_1}} \sin \theta_1
\]

\( \theta_1 = \chi \) between \( d\mathbf{x} \) and \( \mathbf{R_1} \)

\[
d\mathbf{H} \times \hat{R_2} \text{ is into page here}
\]

Induced mag field \( \overrightarrow{H} \) forms loops around the wire.
Biot-Savart Law 2.

calculation of $\vec{H}$ due to a current carrying wire -

the geometry:

the wire $\vec{r}$

$\overrightarrow{dl} = \hat{z} \, dz$

$\vec{r} = r \hat{r} + z \hat{z}$ a vector quantity

$\overrightarrow{dl} \times \vec{r} = (\hat{z} \, dz) \times (r \hat{r} + z \hat{z})$

$= dz \left( r \hat{z} \times \hat{r} + z \hat{z} \times \hat{z} \right) = dz \hat{r} \times \hat{r}$

$= 0$ as $\hat{r}$ is a normal (about z-axis)

vector

$= \phi$ as normal (about z-axis)

$\frac{d\vec{H}}{d\phi} = \frac{I}{4\pi} \frac{\overrightarrow{dl} \times \vec{r}}{1 \rho^3}$

$\rho = \sqrt{(r^2 + z^2)}$

[into page at P]

$= \frac{I \, r \, dz \, \phi}{4\pi \, (r^2 + z^2)^{3/2}} = d\vec{H}$
Biot-Savart Law 3

\[ \vec{H} \] due to current carrying wire

\[ \vec{d}H = \frac{I}{4\pi} \frac{r \, dr \, d\theta}{(r^2 + z^2)^{3/2}} \]

two ways to integrate \( \vec{d}H \)

1. Convert to angles, since \( dz \) a function of \( \theta \)
   
   remember \( \theta \) is between \( d\vec{l} \) and \( \vec{R} \)
   
   \[ \frac{r}{Q} = \sin \theta \]
   \[ R = r \csc \theta \]
   
   \[ z = r \cot \theta \Rightarrow dz = r \frac{d}{d\theta} (\cot \theta) = -r \csc^2 \theta \, d\theta \]

   \[ H = \frac{I}{4\pi} \int_{\theta_1}^{\theta_2} \frac{r \, dr}{(r^2 + z^2)^{3/2}} = \frac{I}{4\pi} \int_{\theta_1}^{\theta_2} \frac{r \csc^2 \theta \, d\theta}{r^2 \csc^2 \theta} \]

   \[ H = \frac{I}{4\pi} \int_{\theta_1}^{\theta_2} \frac{\theta_2}{\theta_1} \, d\theta = \frac{I}{4\pi} \left[ \theta_2 \right]_{\theta_1}^{\theta_2} = \frac{I}{4\pi} \left[ \frac{\theta_2}{\sin \theta} \right]_{\theta_1}^{\theta_2} \]

   \[ H = \frac{I}{4\pi} \left[ \sin \theta_1 \theta_2 - \sin \theta_2 \theta_1 \right] \]

   \[ \text{for} \, l \rightarrow \infty, \, l = l_1 + l_2 \, - \, \text{infinitely long wire} \]

   \[ \text{then} \, \theta_2 \rightarrow \pi, \, \theta_1 \rightarrow 0 \, \Rightarrow \, \sin \theta_1 \theta_2 = z \]

   \[ H = \frac{I}{2\pi l} \vec{r} \left( \sin \theta_1 - \sin \theta_2 \right) \text{ field circles around the wire} \]
2. 2nd way to integrate \( \int dH \)

\[
\vec{H} = \frac{i}{4\pi} \int \frac{d^2 \mathbf{A}}{(r^2 + z^2)^{3/2}} = \text{independent of } z
\]

\[
= \frac{i \Gamma \phi}{4 \pi} \left[ \frac{2}{r^2 \sqrt{r^2 + z^2}} \right]_{-l_i}^{+l_2}
\]

\[
\vec{H} = \frac{i}{4\pi r} \hat{\phi} \left[ \frac{l_2}{\sqrt{r^2 + l_2^2}} + \frac{l_1}{\sqrt{r^2 + l_1^2}} \right]
\]

\[
\text{but } \frac{l_2}{\sqrt{r^2 + l_2^2}} = \cos(\pi - \theta_2) = -\cos \theta_2
\]

\[
\frac{l_1}{\sqrt{r^2 + l_1^2}} = \cos \theta_1
\]

\[
\therefore \frac{H}{4\pi r} = \hat{\phi} \left[ \cos \theta_1 - \cos \theta_2 \right] \text{ as before.}
\]
Magnetic Field of Long Conductor

\[ \mathbf{B} = \hat{\phi} \frac{\mu_0 I}{2\pi r} \]  
(infinitely long wire).
Example 5-3: Magnetic Field of a Loop

Magnitude of field due to dl is

\[ dH = \frac{I}{4\pi R^2} |dl \times \hat{R}| = \frac{I \ dl}{4\pi (a^2 + z^2)} \]

dH is in the r–z plane, and therefore it has components dHr and dHz

z-components of the magnetic fields due to dl and dl’ add because they are in the same direction, but their r-components cancel

Hence for element dl:

\[ dH = \hat{z} dH_z = \hat{z} dH \cos \theta = \hat{z} \frac{I \cos \theta}{4\pi (a^2 + z^2)} \ dl \]

Figure 5-12: Circular loop carrying a current \( I \) (Example 5-3).
Example 5-3: Magnetic Field of a Loop (cont.)

For the entire loop:

\[
H = \hat{\mathbf{z}} \frac{I \cos \theta}{4\pi (a^2 + z^2)} \oint dl = \hat{\mathbf{z}} \frac{I \cos \theta}{4\pi (a^2 + z^2)} (2\pi a). \tag{5.33}
\]

Upon using the relation \( \cos \theta = a/(a^2 + z^2)^{1/2} \), we obtain

\[
H = \hat{\mathbf{z}} \frac{Ia^2}{2(a^2 + z^2)^{3/2}} \text{ (A/m).} \tag{5.34}
\]

At the center of the loop \( (z = 0) \), Eq. (5.34) reduces to

\[
H = \hat{\mathbf{z}} \frac{I}{2a} \text{ (at } z = 0), \tag{5.35}
\]

and at points very far away from the loop such that \( z^2 \gg a^2 \), Eq. (5.34) simplifies to

\[
H = \hat{\mathbf{z}} \frac{Ia^2}{2|z|^3} \text{ (at } |z| \gg a). \tag{5.36}
\]

*Figure 5-12: Circular loop carrying a current \( I \) (Example 5-3).*
Magnetic Field due to Current Carrying Loop, \( \mathbf{l} \)

\[ \mathbf{dl} = r \, dq \, \hat{\phi} \]

\[ \mathbf{\hat{r}} = -r \mathbf{\hat{r}}^2 + z \hat{z} \]  
points from \( \mathbf{dl} \) to \( \mathbf{P} \)

\[ \mathbf{dl} \times \mathbf{\hat{r}} = (r \, dq \, \hat{\phi}) \times (-r \mathbf{\hat{r}}^2 + z \hat{z}) \]

\[ = r \, dq \left[ -r \hat{\phi} \hat{r}^2 + z \hat{z} \right] \]

\[ = -2 \hat{z} \]

\[ \therefore \quad \mathbf{dl} \times \mathbf{\hat{r}} = r \, dq \left[ r \hat{z} + z \hat{\phi} \right] \]
Magnetic Field Due to Current Loop

\[ \mathbf{dH} = \frac{I}{4\pi} \frac{\mathbf{dl} \times \mathbf{r}}{|\mathbf{r}|^3} \quad \text{look at rz plane} \]

\[ |\mathbf{r}| = \sqrt{x^2 + z^2} \]

\[ \mathbf{r} \] components cancel on opp. sides of loop!

\[ \Rightarrow \quad \frac{1}{2} \text{ looped on } z \text{ axis} \]

\[ \mathbf{dH} = \frac{I}{4\pi} \frac{[r \hat{a}_x z + r \hat{a}_z x]}{(r^2 + z^2)^{3/2}} \]

\[ \mathbf{H} = \frac{I}{4\pi} \int_0^{2\pi} \frac{r \hat{a}_z \frac{x}{r}}{(r^2 + z^2)^{3/2}} \quad \int_0^{2\pi} \frac{dx}{r} = 2\pi \]

\[ \mathbf{H} = \frac{I}{2} \frac{2}{z(r^2 + z^2)^{1/2}} \quad \text{at loop center } z=0 \]

\[ \mathbf{H} = \frac{I}{2} \frac{z}{(r^2 + z^2)^{1/2}} \]

**NOTE:** For thin loop, \( z \gg r \Rightarrow \)

\[ \mathbf{H} = \frac{2I}{2} \frac{z}{z^2} = \boxed{\frac{I}{z}} \quad \text{a "mag dipole"} \]

Remember electric field \( \mathbf{E} \times \frac{1}{(d\omega t)^3} \)
Because a circular loop exhibits a magnetic field pattern similar to the electric field of an electric dipole, it is called a magnetic dipole.
Magnetic Field inside Solenoid (max).

I out of page here

Assume length \( l = h + l_z \) of \( N \) turns of wire.

Radius of loop = \( r \) carrying current \( I \)

Inside \( I \) and on axis: \( H = \frac{I}{2} \left( \frac{r^2}{(r^2+z^2)^{3/2}} \right) \) where \( I' = \frac{I}{n} \) current in each

Treat increment of length of solenoid as

\( dz \) as an equivalent loop of \( n dz \) turns

where \( n = \# \) turns/length where \( I = I' \) \( n dz \) turns

so current carried by \( n dz \) turns

\( \Rightarrow \ \int dH = nI' dz \left( \frac{r^2}{(r^2+z^2)^{3/2}} \right) \) on the axis from before
Magnetic Field inside Solenoid, \(z\) axis.

\[
\vec{H} = \int \frac{d^2 \mathbf{l}}{4\pi} = \int \frac{d^2 \mathbf{l}}{2(\mathbf{r}^2 + z^2)^{3/2}} \quad \text{on axis}
\]

\[
= \frac{nI\mathbf{r}^2 z}{2} \int \frac{d\mathbf{l}}{(\mathbf{r}^2 + z^2)^{3/2}}
\]

\[
= nI\mathbf{r}^2 z \left[ \frac{z}{\sqrt{\mathbf{r}^2 + z^2}} \right] \quad \text{field on axis, away from ends}
\]

\[
\vec{H} = \frac{nI\mathbf{r}^2 z}{2} \left[ \frac{d\mathbf{l}}{\sqrt{\mathbf{r}^2 + z^2}} + \frac{d\mathbf{l}}{\sqrt{\mathbf{r}^2 + z^2}} \right]
\]

If \(l_1 \ll l_2 \gg r\), a long solenoid.

\[
\text{then } \vec{H} = \frac{nI\mathbf{r}^2}{2} \quad n = \#\text{ turns/length} \Rightarrow \frac{nIz}{2\text{ total length}}
\]

If we look at field on axis far away from solenoid (ie \(z > r\)), \(l_1, l_2\) then field looks like a current loop with \(nI\) amp-turns at falls off like loop as \(\frac{1}{z^2}\)

\[
\text{what is the mag field at end of solenoid?}
\]

Inside, \(\vec{H}\) is uniform in \(z\) direction.
Magnetic Field of Solenoid. On axis 3.

at end.

\[ l = l_1 + l_2 \]

If \( \vec{H} \) in center of long solenoid is

\[ \vec{H}_{\text{center}} = \vec{H}_{1/2} + \vec{H}_{2/2} \]

where \( \vec{H}_{1/2} \) is free field due to \( 1/2 \) the solenoid

then \( \vec{H}_{1/2} \) is field at the end.

The field on the axis of the solenoid

at the end is just \( 1/2 \) that of the

field in the center.

\[ \vec{H}_{\text{end}} = \frac{1}{2} 2\pi n I \]
Magnetic Dipole

Because a circular loop exhibits a magnetic field pattern similar to the electric field of an electric dipole, it is called a magnetic dipole.
Biot–Savart Law.

Calculate magnetic field $\vec{B}$ at any point of a pre-shaped loop.

\[ \vec{d}B = \frac{I}{4\pi} \frac{\vec{d}l \times \vec{R}}{R^3} = \frac{I}{4\pi} \frac{\vec{d}l \times \vec{R}}{1R^3} \]

1. For horizontal wire, since magnetic fields circulate about wire, no contribution at 0 from either length.

2. For the straight radial segments, $\overline{OA}$, $\overline{OC}$
   \[ \frac{\vec{d}l}{\overrightarrow{AO}} = \frac{\vec{d}l}{\overrightarrow{OC}} \]
   \[ \vec{d}l \parallel \vec{R} \parallel \overrightarrow{00} \]
   \[ \vec{d}l \cdot \vec{R} = 0 \text{ for both segments} \]

3. For the arc $\overline{AC}$
   \[ \frac{\vec{d}l}{\overrightarrow{AC}} = \frac{\vec{d}l}{\overrightarrow{OC}} \times \frac{2 \overrightarrow{OC}}{2} = \frac{\vec{d}l}{\overrightarrow{OC}} \times \frac{\overrightarrow{OC}}{\overrightarrow{OC}} \]
   \[ \vec{d}l = \frac{2}{\overrightarrow{OC}} \frac{\vec{d}l}{\overrightarrow{AC}} \]

\[ \vec{H} = \frac{I}{4\pi} \frac{\overrightarrow{Q} \times \frac{2}{\overrightarrow{OC}}}{2} = \frac{I}{4\pi} \frac{2}{\alpha} \]

$\overrightarrow{Q} = 2\overrightarrow{R}$, complete loop then we get answer from before.
Forces on Parallel Conductors

Note: $B = \mu H$

$$B_1 = -\hat{x} \frac{\mu_0 I_1}{2\pi d}.$$  \hspace{1cm} (5.39)

The force $F_2$ exerted on a length $l$ of wire $I_2$ due to its presence in field $B_1$ may be obtained by applying Eq. (5.12):

$$F_2 = I_2 l \hat{z} \times B_1 = I_2 l \hat{z} \times (-\hat{x}) \frac{\mu_0 I_1}{2\pi d}$$

$$= -\hat{y} \frac{\mu_0 I_1 I_2 l}{2\pi d},$$  \hspace{1cm} (5.40)

and the corresponding force per unit length is

$$F'_2 = \frac{F_2}{l} = -\hat{y} \frac{\mu_0 I_1 I_2}{2\pi d}.$$  \hspace{1cm} (5.41)

A similar analysis performed for the force per unit length exerted on the wire carrying $I_1$ leads to

$$F'_1 = \hat{y} \frac{\mu_0 I_1 I_2}{2\pi d}.$$  \hspace{1cm} (5.42)

Parallel wires attract if their currents are in the same direction, and repel if currents are in opposite directions.
Tech Brief 10: Electromagnets

Figure TF10-1: Solenoid and horseshoe magnets.
Magnetic Levitation

(a) Maglev train

(b) Internal workings of the Maglev train

Figure TF10-5: Magnetic trains. (Courtesy Shanghai.com.)
Maxwell’s Equations

\[ \nabla \cdot \mathbf{D} = \rho_v, \]
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \]
\[ \nabla \cdot \mathbf{B} = 0, \]
\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \]

Under *static* conditions, none of the quantities appearing in Maxwell’s equations are functions of time (i.e., \( \partial / \partial t = 0 \)). *This happens when all charges are permanently fixed in space,* or, *if they move,* *they do so at a steady rate so that* \( \rho_v \) *and* \( \mathbf{J} \) *are constant in time.* Under these circumstances, the time derivatives of \( \mathbf{B} \) and \( \mathbf{D} \) in Eqs. (4.1b) and (4.1d) vanish, and Maxwell’s equations reduce to

**Electrostatics**

\[ \nabla \cdot \mathbf{D} = \rho_v, \quad (4.2a) \]
\[ \nabla \times \mathbf{E} = 0. \quad (4.2b) \]

**Magnetostatics**

\[ \nabla \cdot \mathbf{B} = 0, \quad (4.3a) \]
\[ \nabla \times \mathbf{H} = \mathbf{J}. \quad (4.3b) \]

*Electric and magnetic fields become decoupled under static conditions.*
Ampere's Law

From Maxwell's equations, static case

\[ \nabla \times \mathbf{E} = 0 \]

using Stokes' Theorem we get

\[ \int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{s} = \oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \] conservative field.

Similarly, in magnetostatics:

\[ \nabla \times \mathbf{H} = \mathbf{J} \]

\[ \Rightarrow \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \oint_C \mathbf{H} \cdot d\mathbf{l} = \oint_C \mathbf{J} \cdot d\mathbf{l} = \oint_C \mathbf{M} \cdot d\mathbf{l} \]

\[ = I_{\text{enclosed}} \]

\[ \Rightarrow \oint_C \mathbf{H} \cdot d\mathbf{l} = I_{\text{enclosed}} \] Ampere's law, 1820
Ampère’s Law

\[ \nabla \times \mathbf{H} = \mathbf{J} \quad \leftrightarrow \quad \oint_{C} \mathbf{H} \cdot d\ell = I \]

The sign convention for the direction of the contour path \( C \) in Ampère’s law is taken so that \( I \) and \( \mathbf{H} \) satisfy the right-hand rule defined earlier in connection with the Biot–Savart law. That is, if the direction of \( I \) is aligned with the direction of the thumb of the right hand, then the direction of the contour \( C \) should be chosen along that of the other four fingers.

Note: \( I \) is the current enclosed by the line contour

**Figure 5-16:** Ampère’s law states that the line integral of \( \mathbf{H} \) around a closed contour \( C \) is equal to the current traversing the surface bounded by the contour. This is true for contours (a) and (b), but the line integral of \( \mathbf{H} \) is zero for the contour in (c) because the current \( I \) (denoted by the symbol \( \bigcirc \)) is not enclosed by the contour \( C \).
Internal Magnetic Field of Long Conductor

For $r < a$

\[
\oint_{C_1} \mathbf{H}_1 \cdot d\mathbf{l}_1 = I_1,
\]

Eqn. 5.48

The current $I_1$ flowing through the area enclosed by $C_1$ is equal to the total current $I$ multiplied by the ratio of the area enclosed by $C_1$ to the total cross-sectional area of the wire:

\[
I_1 = \left( \frac{\pi r_1^2}{\pi a^2} \right) I = \left( \frac{r_1}{a} \right)^2 I.
\]

Equating both sides of Eq. (5.48) and then solving for $\mathbf{H}_1$ yields

\[
\mathbf{H}_1 = \hat{\phi} H_1 = \hat{\phi} \frac{r_1}{2\pi a^2} I \quad \text{for } r_1 \leq a.
\] (5.49)
External Magnetic Field of Long Conductor

For $r > a$

(b) For $r = r_2 \geq a$, we choose path $C_2$, which encloses all the current $I$. Hence, $H_2 = \hat{\phi} H_2$, $d\ell_2 = \hat{\phi} r_2 \, d\phi$, and

\[
\oint_{C_2} H_2 \cdot d\ell_2 = 2\pi r_2 H_2 = I,
\]

which yields

\[
H_2 = \hat{\phi} H_2 = \hat{\phi} \frac{I}{2\pi r_2} \quad (\text{for } r_2 \geq a). \quad (5.49b)
\]

Just like with the Biot-Savart law, only easier
Magnetic Field of Toroid

Applying Ampere’s law over contour C:

\[ \oint_C \mathbf{H} \cdot d\ell = I \]

Ampere’s law states that the line integral of \( \mathbf{H} \) around a closed contour \( C \) is equal to the current traversing the surface bounded by the contour.

\[ \oint_C \mathbf{H} \cdot d\ell = \int_0^{2\pi} (\hat{\phi} \mathbf{H}) \cdot \hat{r} \, d\phi = -2\pi r H = -NI. \]

Hence, \( H = NI/(2\pi r) \) and

\[ \mathbf{H} = -\hat{\phi} H = -\hat{\phi} \frac{NI}{2\pi r} \quad \text{for } a < r < b. \]

The magnetic field outside the toroid is zero. Why?
Magnetic Field due to a Current Sheet

Example: 5.6

"Surface" current density
\[ J_s = x J_s \]
\[ \text{current length} \]
\[ \text{current out of page} \]

What is direction of mag field above and below sheet?

For \( z > 0 \), \( \vec{H} = -\hat{z} \vec{H} \)
For \( z < 0 \), \( \vec{H} = +\hat{z} \vec{H} \)

\( \vec{H} \) in a single wire
Magnetic Vector Potential $\mathbf{A}$

**Electrostatics**

\[ \mathbf{E} = -\nabla V \]

\[ \nabla^2 V = -\frac{\rho_v}{\varepsilon} \]

\[ V = \frac{1}{4\pi \varepsilon} \int_{V'} \frac{\rho_v}{R'} \, dV' \]

**Magnetostatics**

\[ \mathbf{B} = \nabla \times \mathbf{A} \quad (\text{Wb/m}^2), \]

\[ \nabla^2 \mathbf{A} = -\mu \mathbf{J} \]

\[ \mathbf{A} = \frac{\mu}{4\pi} \int_{V'} \frac{\mathbf{J}}{R'} \, dV' \quad (\text{Wb/m}). \]
Vector Magnetic Potential

electrostatics, \( \mathbf{E} = -\nabla V \) scalar elec. potential

define \( \mathbf{B} \) in terms of a magnetic potential
such that \( \nabla \cdot \mathbf{B} = 0 \) // Maxwell.

\[ \mathbf{B} = \nabla \times \mathbf{A} \] the vector potential

from Maxwell, \( \nabla \times \mathbf{H} = \mathbf{J} \), and \( \mathbf{B} = \mu \mathbf{A} \)

\[ \nabla \times \mathbf{B} = \mu \mathbf{J} \]

\[ \nabla \times (\nabla \times \mathbf{A}) = \mu \mathbf{J} \]

but for any vector, \( \mathbf{A} \)

\[ \nabla \cdot \nabla \mathbf{A} = \nabla \cdot (\nabla \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) \]

\[ \nabla \cdot \nabla \mathbf{A} = -\nabla \times (\nabla \times \mathbf{A}) = -\mu \mathbf{J} \]

In electrostatics we chose where we had \( V = 0 \).
Here we chose \( \mathbf{A} \) such that \( \nabla \cdot \mathbf{A} = 0 \)
still satisfies \( \mathbf{B} = \nabla \times \mathbf{A} \)

\[ \nabla \cdot (\nabla \times \mathbf{A}) = -\mu \mathbf{J} \] vector Poisson's eqn.

define mag flux \( \Phi = \int \mathbf{B} \cdot d\mathbf{S} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint \mathbf{A} \cdot d\mathbf{l} = \Phi \)
Magnetic Properties of Materials

The magnetic behavior of a material is governed by the interaction of the magnetic dipole moments of its atoms with an external magnetic field. The nature of the behavior depends on the crystalline structure of the material and is used as a basis for classifying materials as diamagnetic, paramagnetic, or ferromagnetic.

\[
B = \mu_0 H + \mu_0 M = \mu_0 (H + M)
\]

\[
M = \chi_m H
\]

\[
B = \mu_0 (H + \chi_m H) = \mu_0 (1 + \chi_m) H,
\]

\[
B = \mu H,
\]
Electron Orbital and Spin Magnetic Moments

Figure 5-20: An electron generates (a) an orbital magnetic moment $m_0$ as it rotates around the nucleus and (b) a spin magnetic moment $m_s$, as it spins about its own axis.
Table 5-2: Properties of magnetic materials.

<table>
<thead>
<tr>
<th></th>
<th>Diamagnetism</th>
<th>Paramagnetism</th>
<th>Ferromagnetism</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permanent magnetic</td>
<td>No</td>
<td>Yes, but weak</td>
<td>Yes, and strong</td>
</tr>
<tr>
<td>dipole moment</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Primary magnetization</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>mechanism</td>
<td>Electron orbital magnetic moment</td>
<td>Electron spin magnetic moment</td>
<td>Magnetized domains</td>
</tr>
<tr>
<td>Direction of induced</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>magnetic field</td>
<td>Opposite</td>
<td>Same</td>
<td>Hysteresis</td>
</tr>
<tr>
<td>(relative to external</td>
<td></td>
<td></td>
<td>(see Fig. 5-22)</td>
</tr>
<tr>
<td>field)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Common substances</td>
<td>Bismuth, copper, diamond, gold, lead, mercury, silver, silicon</td>
<td>Aluminum, calcium, chromium, magnesium, niobium, platinum, tungsten</td>
<td>Iron, nickel, cobalt</td>
</tr>
<tr>
<td>Typical value of $\chi_m$</td>
<td>$\approx -10^{-5}$</td>
<td>$\approx 10^{-5}$</td>
<td>$</td>
</tr>
<tr>
<td>Typical value of $\mu_r$</td>
<td>$\approx 1$</td>
<td>$\approx 1$</td>
<td>$</td>
</tr>
</tbody>
</table>

Thus, $\mu_r \approx 1$ or $\mu \approx \mu_0$ for diamagnetic and paramagnetic substances, which include dielectric materials and most metals. In contrast, $|\mu_r| \gg 1$ for ferromagnetic materials; $|\mu_r|$ of purified iron, for example, is on the order of $2 \times 10^5$. 
Magnetic Hysteresis

(a) Unmagnetized domains
(b) Magnetized domains

Figure 5-22: Typical hysteresis curve for a ferromagnetic material.
Magnetic Domains
Boundary Conditions

By analogy, application of Gauss’s law for magnetism, as expressed by Eq. (5.44), leads to the conclusion that

$$\oint_S \mathbf{B} \cdot ds = 0 \quad \Rightarrow \quad B_{1n} = B_{2n}. \quad (5.79)$$

Thus the normal component of $\mathbf{B}$ is continuous across the boundary between two adjacent media.

Surface currents can exist only on the surfaces of perfect conductors and superconductors. Hence, at the interface between media with finite conductivities, $J_s = 0$ and

$$\hat{n}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s.$$

$$\oint_S \mathbf{D} \cdot ds = Q \quad \Rightarrow \quad D_{1n} - D_{2n} = \rho_s. \quad (5.78)$$
Inside the solenoid:

\[
\mathbf{B} \approx \hat{\mu}_n I = \frac{\hat{\mu}_N I}{l}
\]

(long solenoid with \(l/a \gg 1\))
**Inductance**

**Magnetic Flux**

\[ \Phi = \int_{S} \mathbf{B} \cdot d\mathbf{s} \quad \text{(Wb).} \]

**Flux Linkage**

\[ \Lambda = N \Phi = \mu \frac{N^2}{l} I S \quad \text{(Wb)} \]

**Inductance**

\[ L = \frac{\Lambda}{I} \quad \text{(H).} \]

**Solenoid**

\[ L = \mu \frac{N^2}{l} S \quad \text{(solenoid), \quad (5.95)} \]

and for two-conductor configurations similar to those of Fig. 5-27,

\[ L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{1}{I} \int_{S} \mathbf{B} \cdot d\mathbf{s} \quad \text{(5.96)} \]

-27: To compute the inductance per unit length of inductor transmission line, we need to determine the flux through the area \( S \) between the conductors.
Example 5-7: Inductance of Coaxial Cable

The magnetic field in the region \( S \) between the two conductors is approximately

\[
B = \hat{\Phi} \frac{\mu I}{2\pi r}
\]

Total magnetic flux through \( S \):

\[
\Phi = l \int_{a}^{b} B \, dr = l \int_{a}^{b} \frac{\mu I}{2\pi r} \, dr = \frac{\mu Il}{2\pi} \ln \left( \frac{b}{a} \right)
\]

Inductance per unit length:

\[
L' = \frac{L}{l} = \frac{\Phi}{lI} = \frac{\mu}{2\pi} \ln \left( \frac{b}{a} \right).
\]

Figure 5-28: Cross-sectional view of coaxial transmission line (Example 5-7).
Tech Brief 11: Inductive Sensors

LVDT can measure displacement with submillimeter precision

Figure TF11-1: Linear variable differential transformer (LVDT) circuit.

Figure TF11-2: Amplitude and phase responses as a function of the distance by which the magnetic core is moved away from the center position.
Proximity Sensor

**Figure TF11-5**: Eddy-current proximity sensor.
Magnetic Energy Density

Example 5-8: Magnetic Energy in a Coaxial Cable

Magnetic field in the insulating material is

\[ H = \frac{B}{\mu} = \frac{I}{2\pi r} \]

The magnetic energy stored in the coaxial cable is

\[ W_m = \frac{1}{2} \int \mu H^2 \, dV = \frac{\mu I^2}{8\pi^2} \int \frac{1}{r^2} \, dV \]

\[ w_m = \frac{W_m}{V} = \frac{1}{2} \mu H^2 \quad (J/m^3). \]
Chapter 5 Relationships

Maxwell’s Magnetostatics Equations

Gauss’s Law for Magnetism
\[ \nabla \cdot \mathbf{B} = 0 \quad \iff \quad \oint_{S} \mathbf{B} \cdot d\mathbf{s} = 0 \]

Ampère’s Law
\[ \nabla \times \mathbf{H} = \mathbf{J} \quad \iff \quad \oint_{C} \mathbf{H} \cdot d\mathbf{l} = I \]

Lorentz Force on Charge \( q \)
\[ \mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \]

Magnetic Force on Wire
\[ \mathbf{F}_{m} = I \oint_{C} d\mathbf{l} \times \mathbf{B} \quad (\text{N}) \]

Magnetic Torque on Loop
\[ \mathbf{T} = \mathbf{m} \times \mathbf{B} \quad (\text{N-m}) \]
\[ \mathbf{m} = \hat{\mathbf{n}} N A \quad (\text{A-m}^2) \]

Biot–Savart Law
\[ \mathbf{H} = \frac{I}{4\pi} \int_{l} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \quad (\text{A/m}) \]

Magnetic Field

Infinitely Long Wire
\[ \mathbf{B} = \frac{\dot{\phi}}{2\pi r} \quad (\text{Wb/m}^2) \]

Circular Loop
\[ \mathbf{H} = \frac{Ia^2}{2(a^2 + z^2)^{3/2}} \quad (\text{A/m}) \]

Solenoid
\[ \mathbf{B} \propto \frac{\dot{\mathbf{z}} \mu n I}{l} \quad (\text{Wb/m}^2) \]

Vector Magnetic Potential
\[ \mathbf{B} = \nabla \times \mathbf{A} \quad (\text{Wb/m}^2) \]

Vector Poisson’s Equation
\[ \nabla^2 \mathbf{A} = -\mu \mathbf{J} \]

Inductance
\[ L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{1}{l} \int_{s} \mathbf{B} \cdot d\mathbf{s} \quad (\text{H}) \]

Magnetic Energy Density
\[ w_{m} = \frac{1}{2} \mu H^2 \quad (\text{J/m}^3) \]