EE 135, Winter 2013

Reading: Chapter 3. entire chapter.  
*review of vector algebra and calculus*

Homework will be assigned Thursday.

For laboratory: read lab.2 introduction before lab.

Lecture 6
EE 135. Prelim. #1 / Winter 2013

1. A. The characteristic impedance of a transmission line is:

$$Z_0 = \frac{R' + jωL'}{G' + jωC'}$$

where \( R', L', G', \) and \( C' \) are all "per unit length".
\( \omega \) is the \( \xi \) freq. independent of the line length.

\( Z_0 \) is indep. of line length.

B. \( λ = \frac{c}{f} = \frac{3 \times 10^8 \text{m/sec}}{2.54 \times 10^9 \text{/sec}} = 1.18 \times 10^{-1} \text{m} = λ \)

C. A dispersive line has the phase velocity, \( u_p \), not indep. of the frequency of the signal.

\( R' \) \( u_p = u_p(\omega) \)

Signals in dispersive lines get distorted because different frequency signals travel with different \( u_p \).

D. \( g(4x) = e^{-\left(at-bx\right)^2} \)

ANY function of the form \( f(at±bx) \) describes a wave in one dimension (if \( a, b \) are constants).

\( g(4x) \) describes a wave.

Verify:

\( g(0, x) = e^{-b^2x^2} \) at \( t=0 \) \( \Rightarrow \frac{\partial g}{\partial t} = 0 \) \( \Rightarrow \frac{\partial g}{\partial \tau} = e^{-b^2x^2} \)

\( g(4x) = e^{-\left(at-bx\right)^2} \) at \( \tau = \frac{1}{a} x^2 \) \( \Rightarrow \frac{\partial g}{\partial \tau} = e^{-b^2x^2} \)

Wave moves right for \( \tau = \frac{1}{a} x^2 \) \( \Rightarrow \frac{\partial g}{\partial \tau} = e^{-b^2x^2} \)
EE 135, Prelim. #1 / Winter 2013

1. e/ for all transmission lines,
   \[ \frac{c'}{\sigma'} = \frac{c}{\sigma} \]

\[ \therefore \text{to the extent } \sigma, \sigma' \text{ are independent of } \uparrow, \]
\[ \text{then } \frac{c'}{\sigma'} \text{ don't depend upon up} \]
2. \( z_1 = 1 + 4i \), \( z_2 = 2 + 5i \\
A) \frac{1}{z_1} = \frac{1}{1+4i} = \frac{1-4i}{1+4i} = \frac{1-4i}{1+4i} \cdot \frac{1-4i}{1-4i} = \frac{1-4i}{17} - \frac{4i}{17} = \frac{1}{z_2} \\
\frac{1}{z_2} = \frac{1}{-5+2i} = \frac{1}{-5+2i} \cdot \frac{-5-2i}{-5-2i} = \frac{5+2i}{25-4} = \frac{5}{24} - \frac{2i}{24} = \frac{1}{z_2} \\

B) \mid z_1 z_2 \mid = \mid (1+4i)(-5+2i) \mid = \mid -13-18i \mid \\
= \sqrt{13^2 + 18^2} = 22.2 = \mid z_1 z_2 \mid \\

C) z_1 z_2 = \mid z_1 z_2 \mid e^{j\theta} \text{ where } \theta = \arctan\left(\frac{\text{Im}(z_1 z_2)}{\text{Re}(z_1 z_2)}\right) \\
\text{where } \theta \text{ in } \mathbb{Z}^{2D} \text{ quadrant} \\
\frac{\text{Im}(z_1 z_2)}{\text{Re}(z_1 z_2)} = \frac{-18}{13} = 1.385 \text{, but } x \text{ must be } \text{ in } \mathbb{Z}^{2D} \text{ quadrant} \\
180^\circ + 54.17^\circ = 234.17^\circ \\
\text{or } 222.2 e^{j(234.17^\circ)} \\

D) n(t) = A \sin(wt + \theta) = A \sin\left(wt + \frac{\pi}{2}\right) = \text{Re}\left[ A e^{j(wt + \theta)} \right] \\
n'(t) = \frac{1}{j} \text{Re}\left[ A e^{j(wt + \theta)} \right] = \frac{\partial}{\partial t} \text{Re}\left[ A e^{j(wt + \theta)} \right] = -w^2 A e^{j(wt + \theta)} \\
n''(t) = \frac{\partial^2}{\partial t^2} \text{Re}\left[ A e^{j(wt + \theta)} \right] = w^2 A e^{j(wt + \theta)} = \frac{\partial^2}{\partial t^2} \text{Re}\left[ A e^{j(wt + \theta)} \right] = -j \text{Re}\left[ A e^{j(wt + \theta)} \right] \\
n''(t) = w^2 A e^{j(wt + \theta)} \text{ since } e^{-j\pi/2} = -j
we only consider transform effects if $\frac{L}{\lambda} \leq 10^{-2}$

A. $f = 100\,\text{Hz}$, $l = 10\,\text{cm} \Rightarrow f = 10^4\,\text{Hz}$

$\frac{\lambda}{f} = \frac{3 \times 10^8\,\text{m/s}}{10^4\,\text{Hz}} = 3\times 10^4\,\text{m}$

$\frac{l}{\lambda} = \frac{10^{-2}\,\text{m}}{3\times 10^4\,\text{m}} = \frac{1}{3} \times 10^{-5} \leq 10^{-2}$ NO

B. $f = 200\,\text{GHz}$, $l = 10^2\,\text{mm} \Rightarrow f = 2 \times 10^9\,\text{Hz}$

$\frac{\lambda}{f} = \frac{3 \times 10^8\,\text{m/s}}{2 \times 10^{11}\,\text{Hz}} = 1.5 \times 10^{-3}\,\text{m}$

$\frac{l}{\lambda} = \frac{10^{-2}\,\text{m}}{1.5 \times 10^{-3}\,\text{m}} = \frac{2}{3} \times 10^{-1} > 10^{-2}$ YES

C. $w = 6.28 \times 10^3\,\text{rad/s}$, $l = 5\,\text{km}$

$\Rightarrow f = \frac{w}{2\pi} = 10^5\,\text{Hz}$, $\lambda = 5 \times 10^3\,\text{m}$

$\frac{\lambda}{f} = \frac{1 \times 10^8\,\text{m/s}}{10^5\,\text{Hz}} = 3 \times 10^3\,\text{m}$

$\frac{l}{\lambda} = \frac{5 \times 10^3\,\text{m}}{3 \times 10^3\,\text{m}} = \frac{5}{3} \geq 10^{-2}$ YES

D. $\frac{L}{\lambda}$ independent of load ($R, L, C$): \hline no effect changing load
4.) \( V_2(t) \)

\[ R = 10^2 \Omega, \quad C = 100 \text{pF} = 10^{-12} \text{F} \]

\[ Z_0 = 50 \Omega \]

we assume a constant/small time variation

with \( \times \text{freq} = \omega = 2\pi f / \)

A. \( Z_L = ? \Rightarrow \frac{1}{Z_L} = \frac{1}{R} + \frac{1}{X_C} = \frac{1}{j\omega C} \)

\[ \therefore \frac{1}{Z_L} = \frac{1}{R} + j\omega C \Rightarrow Z_L = \frac{R}{1 + j\omega RC} \]

\[ \therefore Z_L = \frac{R}{1 + j\omega RC} \cdot \frac{1 - j\omega RL}{1 - j\omega RL} \]

\[ \therefore Z_L = \frac{R - j\omega RL}{1 + (\omega RL)^2} \]

\[ RC = 10^2 \times 10^{-12} = 10^{-8} \]

\[ (\omega L)^2 = 10^{-16} \]

**effective impedance of load.**

\[ Z_L \approx R = 10^2 \Omega \approx \text{load at } f = 10 \text{kHz} \]

\[ \therefore I_2 / I_1 = 1 \iff \left[ \frac{Z - Z_0}{Z + Z_0} \right] \text{current reflected} \]

\[ \therefore \text{we can evaluate in terms of } \omega \]
4. Now.

Let evaluate $\Gamma$, the voltage reflection coefficient for general $\omega$.

Let $A = W RC$ then

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{R - jRA}{1 + \frac{j\omega}{1 + \frac{j\omega}{R - jRA}} - Z_0} = \frac{R - Z_0(1 + \omega^2) - jRA}{R + Z_0(1 + \omega^2) - jRA}$$

Let $B = Z_0(1 + \omega^2)$ give

$$\Gamma = \frac{R - B - jRA}{R + B - jRA} \quad \text{and} \quad \frac{R - B - jRA}{R + B - jRA} = \frac{R - B - jRA}{R + B - jRA}$$

$$\Gamma = \frac{R - B - jRA}{R + B - jRA}$$

$$\Gamma = \frac{R - B - jRA}{R + B - jRA}$$

$$\Gamma = \frac{\sqrt{(R_B^2 + (RA)^2)} \cdot e^{-j\theta}}{\sqrt{(R_B^2 + (RA)^2)^2}}$$

where $\theta = \arctan \left( \frac{\text{Im} \Gamma}{\text{Re} \Gamma} \right)$

$$\theta = \arctan \left[ \frac{-2RAB}{R^2 - B^2 + (RA)^2} \right]$$

The current reflected to load is

$$\frac{I^*}{I_0} = \frac{\text{current reflected}}{\text{current incident}} = -\Gamma = -\Gamma e^{j\theta}$$

$$\frac{I^*}{I_0} = \frac{\sqrt{(R^2 - B^2 + (RA)^2)^2 + 4(RAB)^2}}{(R + B)^2 + (RA)^2} = |\Gamma|$$
C) The fraction of time avg. power delivered to load:
\[ F = (1 - |\Gamma|^2) \]
with
\[ |\Gamma|^2 = \left( \frac{R^2 - Z_0^2 + (RM)^2 + 4(RM)^2}{(R+Z_0^2 + (RM)^2} \right) \]

D) The way to increase time avg. power is to reduce |\Gamma|

Since \[ \Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \]
This means \( Z_L \rightarrow Z_0 \)

Aim for a matched load \( Z_L = Z_0 \)

NOTE: If \( f = 10\text{kHz} \) then \( Z_L \geq R = 10^2 \Omega \) from before

Then \( \Gamma = \left( \frac{R - Z_0(1 + b)^2 - jRB}{R + jZ_0(1 + b)^2 - jRB} \right) \)

\( P = \text{Watts} = 10^{-6} \text{W} \) so \( P \leq 10^5 \text{rad/m} \) \( (f = 10\text{kHz}) \)

Then \( R = 10^{-3} \) \( R_0 = 10^{-1} \) \( 1 + b^2 = 1 \)

\[ \Gamma = \frac{100 - 50 - j\frac{1}{2}}{100 + 50 - j\frac{1}{2}} \leq \frac{\frac{1}{3}}{\frac{1}{3}} = 1 \]

So \( \Gamma^2 = \frac{1}{\frac{1}{9}} = 9 \Rightarrow F = 1 - \frac{1}{9} = 0.89 = F' \)
5. $f = 300 \text{MHz}$, $Z_0 = 50 \Omega$ lossless line, $G_e = 1.33$

For lossless short circuit trans. line,

$$Z_{in} = jX_{in}$$

$$Z_{in}^e = \frac{X_{in}^e}{Z_0}$$

$$l = \frac{1}{\beta} \arctan \left( \frac{X_{in}^e}{Z_0} \right)$$

$$\beta = \frac{\omega}{V_p \text{ phase velocity of line}}$$

$$V_p = \frac{C_0}{\sqrt{\mu}} = \frac{866 \text{ C}}{1.6} = 866 \text{ C}$$

$$\beta = \frac{20f}{866} = \frac{20 \times 3 \times 10^8}{866 \times 3 \times 10^8} = 1.25 \text{ rad/m} = \beta$$

$$l = \frac{1}{1.25/m} \arctan \left( \frac{1.25}{50\Omega} \right) = 0.675 + \pi n, n = 0, 1, 2, \ldots$$

:. shortest length $l = 0.093m = 9.3 \text{ cm} = \ell$

\[ \text{Diagram of circuit elements} \]
3. VECTOR ANALYSIS

Applied EM by Ulaby, Michielssen and Ravaioli
Chapter 3 Overview

<table>
<thead>
<tr>
<th>Chapter Contents</th>
<th>Objectives</th>
</tr>
</thead>
<tbody>
<tr>
<td>Overview, 131</td>
<td>Upon learning the material presented in this chapter, you should be able to:</td>
</tr>
<tr>
<td>3-1 Basic Laws of Vector Algebra, 131</td>
<td>1. Use vector algebra in Cartesian, cylindrical, and spherical coordinate systems.</td>
</tr>
<tr>
<td>3-2 Orthogonal Coordinate Systems, 137</td>
<td>2. Transform vectors between the three primary coordinate systems.</td>
</tr>
<tr>
<td>3-3 Transformations Between Coordinated Systems, 146</td>
<td>3. Calculate the gradient of a scalar function and the divergence and curl of a vector function in any of the three primary coordinate systems.</td>
</tr>
<tr>
<td>3-4 Gradient of a Scalar Field, 151</td>
<td>4. Apply the divergence theorem and Stokes’s theorem.</td>
</tr>
<tr>
<td>3-5 Divergence of a Vector Field, 156</td>
<td></td>
</tr>
<tr>
<td>3-6 Curl of a Vector Field, 162</td>
<td></td>
</tr>
<tr>
<td>3-7 Laplacian Operator, 166</td>
<td></td>
</tr>
<tr>
<td>Chapter 3 Relationships, 167</td>
<td></td>
</tr>
<tr>
<td>Chapter Highlights, 168</td>
<td></td>
</tr>
<tr>
<td>Glossary of Important Terms, 168</td>
<td></td>
</tr>
<tr>
<td>Problems, 169</td>
<td></td>
</tr>
</tbody>
</table>
Laws of Vector Algebra

\[ \mathbf{A} = \hat{\mathbf{a}} |\mathbf{A}| = \hat{\mathbf{a}} A \]

\[ \mathbf{A} = \hat{\mathbf{x}} A_x + \hat{\mathbf{y}} A_y + \hat{\mathbf{z}} A_z \]

\[ A = |\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} \]

\[ \hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = \frac{\hat{\mathbf{x}} A_x + \hat{\mathbf{y}} A_y + \hat{\mathbf{z}} A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \]
Properties of Vector Operations

Equality of Two Vectors

\[ \mathbf{A} = \hat{\mathbf{a}} A = \hat{x} A_x + \hat{y} A_y + \hat{z} A_z, \quad (3.6a) \]
\[ \mathbf{B} = \hat{\mathbf{b}} B = \hat{x} B_x + \hat{y} B_y + \hat{z} B_z, \quad (3.6b) \]

then \( \mathbf{A} = \mathbf{B} \) if and only if \( A = B \) and \( \hat{a} = \hat{b} \), which requires that \( A_x = B_x, \ A_y = B_y, \) and \( A_z = B_z. \)

*Equality of two vectors does not necessarily imply that they are identical; in Cartesian coordinates, two displaced parallel vectors of equal magnitude and pointing in the same direction are equal, but they are identical only if they lie on top of one another.*

Commutative property

\[ \mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \]

**Figure 3-3:** Vector addition by (a) the parallelogram rule and (b) the head-to-tail rule.
Position Vector: From origin to point P

\[ \mathbf{R}_1 = \overrightarrow{OP}_1 = \hat{x}x_1 + \hat{y}y_1 + \hat{z}z_1 \]

\[ \mathbf{R}_2 = \overrightarrow{OP}_2 = \hat{x}x_2 + \hat{y}y_2 + \hat{z}z_2 \]

Distance Vector: Between two points

\[ \mathbf{R}_{12} = \overrightarrow{P_1P_2} \]

\[ = \mathbf{R}_2 - \mathbf{R}_1 \]

\[ = \hat{x}(x_2 - x_1) + \hat{y}(y_2 - y_1) + \hat{z}(z_2 - z_1) \]

The distance \( d \) between \( P_1 \) and \( P_2 \) equals the magnitude of \( \mathbf{R}_{12} \):

\[ d = |\mathbf{R}_{12}| \]

\[ = \left[ (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \right]^{1/2}. \quad (3.12) \]
How does a GPS receiver determine its location?
GPS: Minimum of 4 Satellites Needed

Unknown: location of receiver \((x_0, y_0, z_0)\)
Also unknown: time offset of receiver clock \(t_0\)

Quantities known with high precision:
locations of satellites and their atomic clocks (satellites use expensive high precision clocks, whereas receivers do not)

Solving for 4 unknowns requires at least 4 equations (four satellites)

\[
\begin{align*}
    d_1^2 &= (x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 = c \left[(t_1 + t_0)\right]^2, \\
    d_2^2 &= (x_2 - x_0)^2 + (y_2 - y_0)^2 + (z_2 - z_0)^2 = c \left[(t_2 + t_0)\right]^2, \\
    d_3^2 &= (x_3 - x_0)^2 + (y_3 - y_0)^2 + (z_3 - z_0)^2 = c \left[(t_3 + t_0)\right]^2, \\
    d_4^2 &= (x_4 - x_0)^2 + (y_4 - y_0)^2 + (z_4 - z_0)^2 = c \left[(t_4 + t_0)\right]^2.
\end{align*}
\]
Vector Multiplication: Scalar Product or "Dot Product"

\[ \mathbf{A} \cdot \mathbf{B} = AB \cos \theta_{AB} \]

Hence:

\[ A = |A| = \sqrt{\mathbf{A} \cdot \mathbf{A}} \]

\[ \theta_{AB} = \cos^{-1} \left[ \frac{\mathbf{A} \cdot \mathbf{B}}{\sqrt{\mathbf{A} \cdot \mathbf{A}} \sqrt{\mathbf{B} \cdot \mathbf{B}}} \right] \]

\[ \hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1, \]
\[ \hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{x} = 0. \]

If \( \mathbf{A} = (A_x, A_y, A_z) \) and \( \mathbf{B} = (B_x, B_y, B_z) \), then

\[ \mathbf{A} \cdot \mathbf{B} = (\hat{x}A_x + \hat{y}A_y + \hat{z}A_z) \cdot (\hat{x}B_x + \hat{y}B_y + \hat{z}B_z). \]

Hence:

\[ \mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z. \]
Vector Multiplication: Vector Product or "Cross Product"

\[ \mathbf{A} \times \mathbf{B} = \hat{n} \mathbf{A} \mathbf{B} \sin \theta_{\mathbf{A}\mathbf{B}} \]

\[ \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \quad \text{(anticommutative)} \]

\[ \mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \quad \text{(distributive)} \]

\[ \mathbf{A} \times \mathbf{A} = 0 \]

\[ \hat{x} \times \hat{y} = \hat{z}, \quad \hat{y} \times \hat{z} = \hat{x}, \quad \hat{z} \times \hat{x} = \hat{y}. \quad (3.25) \]

Note the cyclic order \((x\, y\, z \, x\, y\, z \ldots)\). Also,

\[ \hat{x} \times \hat{x} = \hat{y} \times \hat{y} = \hat{z} \times \hat{z} = 0. \quad (3.26) \]

If \( \mathbf{A} = (A_x, A_y, A_z) \) and \( \mathbf{B} = (B_x, B_y, B_z) \),

\[ \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \]
Example 3-1: Vectors and Angles

In Cartesian coordinates, vector \( \mathbf{A} \) points from the origin to point \( P_1 = (2, 3, 3) \), and vector \( \mathbf{B} \) is directed from \( P_1 \) to point \( P_2 = (1, -2, 2) \). Find
(a) vector \( \mathbf{A} \), its magnitude \( A \), and unit vector \( \hat{\mathbf{a}} \),
(b) the angle between \( \mathbf{A} \) and the y-axis,
(c) vector \( \mathbf{B} \),
(d) the angle \( \theta_{AB} \) between \( \mathbf{A} \) and \( \mathbf{B} \), and
(e) the perpendicular distance from the origin to vector \( \mathbf{B} \).

Solution: (a) Vector \( \mathbf{A} \) is given by the position vector of \( P_1 = (2, 3, 3) \) as shown in Fig. 3-7. Thus,
\[
\mathbf{A} = \hat{x}2 + \hat{y}3 + \hat{z}3,
\]
\[
A = |\mathbf{A}| = \sqrt{2^2 + 3^2 + 3^2} = \sqrt{22},
\]
\[
\hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = (\hat{x}2 + \hat{y}3 + \hat{z}3)/\sqrt{22}.
\]

(b) The angle \( \beta \) between \( \mathbf{A} \) and the y-axis is obtained from
\[
\mathbf{A} \cdot \hat{\mathbf{y}} = |\mathbf{A}||\hat{\mathbf{y}}| \cos \beta = A \cos \beta,
\]
or
\[
\beta = \cos^{-1} \left( \frac{\mathbf{A} \cdot \hat{\mathbf{y}}}{A} \right) = \cos^{-1} \left( \frac{3}{\sqrt{22}} \right) = 50.2^\circ.
\]

(c) Vector \( \mathbf{B} \),
\[
\mathbf{B} = \hat{x}(1 - 2) + \hat{y}(-2 - 3) + \hat{z}(2 - 3) = -\hat{x} - \hat{y}5 - \hat{z}.
\]

(d) The angle \( \theta_{AB} \) is given by
\[
\theta_{AB} = \cos^{-1} \left( \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} \right) = \cos^{-1} \left( \frac{(-2 - 15 - 3)}{\sqrt{22} \sqrt{27}} \right) = 145.1^\circ.
\]

(e) The perpendicular distance between the origin and vector \( \mathbf{B} \) is the distance \( |\overrightarrow{OP_3}| \) shown in Fig. 3-7. From right triangle \( \overrightarrow{OP_1 P_3} \),
\[
|\overrightarrow{OP_3}| = |\mathbf{A}| \sin(180^\circ - \theta_{AB})
\]
\[
= \sqrt{22} \sin(180^\circ - 145.1^\circ) = 2.68.
\]
Triple Products

Scalar Triple Product

\[ A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B). \]

\[ A \cdot (B \times C) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \]

Example 3-2: Vector Triple Product

Given \( A = \hat{x} - \hat{y} + 2\hat{z}, \ B = \hat{y} + 3\hat{z}, \) and \( C = -2\hat{x} + 2\hat{z}, \) find \((A \times B) \times C\) and compare it with \( A \times (B \times C).\)

Solution:

\[ A \times B = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = -\hat{x}3 - \hat{y} + \hat{z} \]

and

\[ (A \times B) \times C = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -3 & -1 & 1 \\ -2 & 0 & 3 \end{vmatrix} = -\hat{x}3 + \hat{y}7 - \hat{z}2. \]

A similar procedure gives \( A \times (B \times C) = \hat{x}2 + \hat{y}4 + \hat{z}. \)

Vector Triple Product

\[ A \times (B \times C) = B(A \cdot C) - C(A \cdot B), \]

which is known as the “bac-cab” rule.

Hence:

\[ A \times (B \times C) \neq (A \times B) \times C \]
Differential length vector

\[ dl = \hat{x} \, dl_x + \hat{y} \, dl_y + \hat{z} \, dl_z = \hat{x} \, dx + \hat{y} \, dy + \hat{z} \, dz, \quad (3.34) \]

where \( dl_x = dx \) is a differential length along \( \hat{x} \), and similar interpretations apply to \( dl_y = dy \) and \( dl_z = dz \).

Differential area vectors

\[
\begin{align*}
    ds_x &= \hat{x} \, dl_y \, dl_z = \hat{x} \, dy \, dz \quad & \text{(y–z plane)}, \\
    ds_y &= \hat{y} \, dx \, dz \quad & \text{(x–z plane),} \\
    ds_z &= \hat{z} \, dx \, dy \quad & \text{(x–y plane).}
\end{align*}
\]

A differential volume equals the product of all three differential lengths:

\[ dv = dx \, dy \, dz. \quad (3.36) \]
<table>
<thead>
<tr>
<th>Table 3-1: Summary of vector relations.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cartesian Coordinates</strong></td>
</tr>
<tr>
<td>Coordinate variables</td>
</tr>
<tr>
<td>Vector representation ( \mathbf{A} = )</td>
</tr>
<tr>
<td>Magnitude of ( \mathbf{A} ) (</td>
</tr>
<tr>
<td>Position vector ( \overrightarrow{OP} = )</td>
</tr>
<tr>
<td>Base vectors properties</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Dot product ( \mathbf{A} \cdot \mathbf{B} = )</td>
</tr>
</tbody>
</table>
| Cross product \( \mathbf{A} \times \mathbf{B} = \) | \[
\begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
A_x & A_y & A_z \\
B_x & B_y & B_z
\end{vmatrix}
\] | \[
\begin{vmatrix}
\hat{r} & \hat{\phi} & \hat{z} \\
A_r & A_\phi & A_z \\
B_r & B_\phi & B_z
\end{vmatrix}
\] | \[
\begin{vmatrix}
\hat{R} & \hat{\theta} & \hat{\phi} \\
A_R & A_\theta & A_\phi \\
B_R & B_\theta & B_\phi
\end{vmatrix}
\] |
| Differential length \( d\mathbf{l} = \) | \( \hat{x} dx + \hat{y} dy + \hat{z} dz \) | \( \hat{r} dr + \hat{\phi} d\phi + \hat{z} dz \) | \( \hat{R} dR + \hat{\theta} d\theta + \hat{\phi} R \sin \theta \ d\phi \) |
| Differential surface areas | \( ds_x = \hat{x} dy \ dz \) | \( ds_r = \hat{r} d\phi \ dz \) | \( ds_R = \hat{R} R^2 \sin \theta \ d\theta \ d\phi \) |
| | \( ds_y = \hat{y} dx \ dz \) | \( ds_{\phi} = \hat{\phi} dr \ dz \) | \( ds_{\theta} = \hat{\theta} R \sin \theta \ dr \ d\phi \) |
| | \( ds_z = \hat{z} dx \ dy \) | \( ds_z = \hat{z} r \ dr \ d\phi \) | \( ds_{\phi} = \hat{\phi} R \ dr \ d\theta \) |
| Differential volume \( dV = \) | \( dx \ dy \ dz \) | \( r \ dr \ d\phi \ dz \) | \( R^2 \sin \theta \ dr \ d\theta \ d\phi \) |
Cylindrical Coordinate System

The mutually perpendicular base vectors are \( \hat{r}, \hat{\phi}, \) and \( \hat{z}, \) with \( \hat{r} \) pointing away from the origin along \( r, \) \( \hat{\phi} \) pointing in a direction tangential to the cylindrical surface, and \( \hat{z} \) pointing along the vertical. Unlike the Cartesian system, in which the base vectors \( \hat{x}, \hat{y}, \) and \( \hat{z} \) are independent of the location of \( P, \) in the cylindrical system both \( \hat{r} \) and \( \hat{\phi} \) are functions of \( \phi. \)
The base unit vectors obey the following right-hand cyclic relations:

\[ \hat{r} \times \hat{\phi} = \hat{z}, \quad \hat{\phi} \times \hat{z} = \hat{r}, \quad \hat{z} \times \hat{r} = \hat{\phi}, \quad (3.37) \]

and like all unit vectors, \( \hat{r} \cdot \hat{r} = \hat{\phi} \cdot \hat{\phi} = \hat{z} \cdot \hat{z} = 1 \), and
\( \hat{r} \times \hat{r} = \hat{\phi} \times \hat{\phi} = \hat{z} \times \hat{z} = 0 \).

In cylindrical coordinates, a vector is expressed as

\[ \mathbf{A} = \hat{a}|\mathbf{A}| = \hat{r}A_r + \hat{\phi}A_\phi + \hat{z}A_z, \quad (3.38) \]

\[ dl_r = dr, \quad dl_\phi = r \, d\phi, \quad dl_z = dz. \quad (3.41) \]

Note that the differential length along \( \hat{\phi} \) is \( r \, d\phi \), not just \( d\phi \).

The differential length \( dl \) in cylindrical coordinates is given by

\[ dl = \hat{r} \, dr + \hat{\phi} \, d\phi + \hat{z} \, dz = \hat{r} \, dr + \hat{\phi} \, r \, d\phi + \hat{z} \, dz. \quad (3.42) \]
Example 3-3: Distance Vector in Cylindrical Coordinates

Find an expression for the unit vector of vector \( \mathbf{A} \) shown in Fig. 3-11 in cylindrical coordinates.

Solution: In triangle \( OP_1P_2 \),

\[
\overrightarrow{OP_2} = \overrightarrow{OP_1} + \mathbf{A}.
\]

Hence,

\[
\mathbf{A} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \hat{r}r_0 - \hat{z}h,
\]

and

\[
\hat{a} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\hat{r}r_0 - \hat{z}h}{\sqrt{r_0^2 + h^2}}.
\]

We note that the expression for \( \mathbf{A} \) is independent of \( \phi_0 \). That is, all vectors from point \( P_1 \) to any point on the circle defined by \( r = r_0 \) in the \( x-y \) plane are equal in the cylindrical coordinate system. The ambiguity can be eliminated by specifying that \( \mathbf{A} \) passes through a point whose \( \phi = \phi_0 \).
Example 3-4: Cylindrical Area

Find the area of a cylindrical surface described by $r = 5$, $30^\circ \leq \phi \leq 60^\circ$, and $0 \leq z \leq 3$ (Fig. 3-12).

Solution: The prescribed surface is shown in Fig. 3-12. Use of Eq. (3.43a) for a surface element with constant $r$ gives

$$S = r \int_{\phi=30^\circ}^{60^\circ} d\phi \int_{z=0}^{3} dz$$

$$= 5\phi \left|_{\pi/6}^{\pi/3} \right|_{0}^{3}$$

$$= \frac{5\pi}{2}.$$  

Note that $\phi$ had to be converted to radians before evaluating the integration limits.

Figure 3-12: Cylindrical surface of Example 3-4.
Spherical Coordinate System

\[ \hat{R} \times \hat{\theta} = \hat{\phi}, \quad \hat{\theta} \times \hat{\phi} = \hat{R}, \quad \hat{\phi} \times \hat{R} = \hat{\theta}. \]  \hspace{1cm} (3.45)

A vector with components \( A_R, A_\theta, \) and \( A_\phi \) is written as

\[ \mathbf{A} = \hat{a} |\mathbf{A}| = \hat{R}A_R + \hat{\theta}A_\theta + \hat{\phi}A_\phi, \]  \hspace{1cm} (3.46)

and its magnitude is

\[ |\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \sqrt{A_R^2 + A_\theta^2 + A_\phi^2}. \]  \hspace{1cm} (3.47)

The position vector of point \( P = (R_1, \theta_1, \phi_1) \) is simply

\[ \mathbf{R}_1 = \overrightarrow{OP} = \hat{R}R_1, \]  \hspace{1cm} (3.48)
Let's look at the 3 diff' quantites in sph. coord.

diff' length, area, volume

**diff' length (a vector)**

\[ dl = Rd\theta \hat{\theta} + \hat{\phi} d\phi + \hat{\mathbf{r}} dr \]

\[ = Rd\theta \hat{\theta} \]

\[ = Rd\theta \]

\[ = R \sin \theta \phi \]

\[ \therefore dl = Rd\theta \hat{\theta} + \hat{\phi} d\phi + \hat{\mathbf{r}} R \sin \theta \phi \]

as before a vector in sph. coord is:

\[ \hat{\mathbf{A}} = R \hat{A}_r + \hat{\theta} (\theta \hat{A}_\theta + \hat{\phi} A_\phi) \]

\[ |\hat{\mathbf{A}}| = \sqrt{A_r^2 + A_\theta^2 + A_\phi^2} \]

**the diff' surface - a vector**

\[ dS_r = Rd\theta d\phi \hat{\mathbf{r}} \]

\[ = \hat{\mathbf{r}} (Rd\theta)(R \sin \theta \phi) \]

\[ dS_r = \hat{\mathbf{r}} (R^2 \sin \theta \phi) \]

\[ dS_\theta = \hat{\theta} d\phi dl_r \]

\[ = \hat{\theta} (R \sin \theta \phi)(dr) \]

\[ dS_\phi = \hat{\phi} dl_r d\theta \]

\[ = \hat{\phi} (dr)(Rd\theta) \]

\[ dS = dS_r + dS_\theta + dS_\phi \]

**be careful how this is drawn**
an example of calculating surf area in spherical words// consider spherical strip of radius = a
bounded by angle θ₁, θ₂ as shown

\[ dS_r = \hat{r} \, d\theta \, \hat{\phi} \]
\[ = \hat{r} \, (ad\theta) \, (as\hat{\phi}) \]
\[ \therefore \frac{dS}{S} = \hat{r} \, dS_r \]
\[ \therefore S = \int_{\theta_1}^{\theta_2} \, a^2 \, \sin \theta \, d\theta \]

\[ S = a^2 \left[ -\cos \theta \right]_{\theta_1}^{\theta_2} \]

\[ S = a^2 \, 2\pi \left( \sin \theta_1 - \sin \theta_2 \right) \]

pointing radially inward

Trick is to carefully draw the geometry so you know what the differences are //
Finally the differential volume (as a scalar)

\[ dV = dr \, d\theta \, d\phi \]

\[ = (dr)(r \, d\theta)(r \sin \theta \, d\phi) \]

\[ dV = r^2 \sin \theta \, dr \, d\theta \, d\phi \]

Now we have just given three orthogonal unit systems, what are the relationships between them? (Part of Lab 1 Exercise)

- Rect \( \rightarrow \) Spherical
- \( x, y, z \rightarrow r, \theta, \phi \)

\[ R = \sqrt{x^2 + y^2 + z^2} \]

\[ \theta = \arctan \left( \frac{\sqrt{x^2 + y^2}}{z} \right) \]

\[ \phi = \arctan \left( \frac{y}{x} \right) \]

Similarly

\[ x = (r \sin \theta) \sin \phi \]

\[ y = (r \sin \theta) \cos \phi \]

\[ z = r \cos \theta \]

The transformation between these unit systems is given in Table 3.2 (pg 117).

Leave it to the student to do Rect \( \rightarrow \) Cyl / Rect \( \rightarrow \) Sph with respect to differential operators.
Example 3-5: Surface Area in Spherical Coordinates

The spherical strip shown in Fig. 3-15 is a section of a sphere of radius 3 cm. Find the area of the strip.

![Spherical strip of Example 3-5.](image)

**Solution:** Use of Eq. (3.50b) for the area of an elemental spherical area with constant radius \(R\) gives

\[
S = R^2 \int_{\theta = 30^\circ}^{60^\circ} \sin \theta \, d\theta \int_{\phi = 0}^{2\pi} \, d\phi
\]

\[
= 9(-\cos \theta)^{60^\circ}_{30^\circ} \int_{\phi = 0}^{2\pi} \, d\phi
\]

\[
= 18\pi(\cos 30^\circ - \cos 60^\circ) = 20.7 \text{ cm}^2.
\]

Example 3-6: Charge in a Sphere

A sphere of radius 2 cm contains a volume charge density \(\rho_v\) given by

\[
\rho_v = 4\cos^2 \theta \quad (\text{C/m}^3).
\]

Find the total charge \(Q\) contained in the sphere.

**Solution:**

\[
Q = \int \rho_v \, dV
\]

\[
= \int_{\phi = 0}^{2\pi} \int_{\theta = 0}^{\pi} \left(4\cos^2 \theta \right) R^2 \sin \theta \, dR \, d\theta \, d\phi
\]

\[
= 4 \int_{\phi = 0}^{2\pi} \int_{\theta = 0}^{\pi} \left(\frac{R^3}{3}\right) \sin \theta \cos^2 \theta \, d\theta \, d\phi
\]

\[
= \frac{32}{3} \times 10^{-6} \int_{\phi = 0}^{2\pi} \left(\frac{-\cos^3 \theta}{3}\right) \Bigg|_{0}^{\pi} \, d\phi
\]

\[
= \frac{64}{9} \times 10^{-6} \int_{0}^{2\pi} \, d\phi
\]

\[
= \frac{128\pi}{9} \times 10^{-6} = 44.68 \quad (\mu\text{C}).
\]
Technology Brief 5: GPS

How does a GPS receiver determine its location?

Figure TF5-1: iPhone map feature.

Figure TF5-2: GPS nominal satellite constellation. Four satellites in each plane, 20,200 km altitudes, 55° inclination.
**GPS: Minimum of 4 Satellites Needed**

Unknown: location of receiver \((x_0, y_0, z_0)\)

Also unknown: time offset of receiver clock \(t_0\)

Quantities known with high precision:
locations of satellites and their atomic clocks (satellites use expensive high precision clocks, whereas receivers do not)

Solving for 4 unknowns requires at least 4 equations (four satellites)

\[
\begin{align*}
    d_1^2 &= (x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 = c [(t_1 + t_0)]^2, \\
    d_2^2 &= (x_2 - x_0)^2 + (y_2 - y_0)^2 + (z_2 - z_0)^2 = c [(t_2 + t_0)]^2, \\
    d_3^2 &= (x_3 - x_0)^2 + (y_3 - y_0)^2 + (z_3 - z_0)^2 = c [(t_3 + t_0)]^2, \\
    d_4^2 &= (x_4 - x_0)^2 + (y_4 - y_0)^2 + (z_4 - z_0)^2 = c [(t_4 + t_0)]^2.
\end{align*}
\]
To solve a problem, we select the coordinate system that best fits its geometry.

Sometimes we need to transform between coordinate systems.

\[ r = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1}\left(\frac{y}{x}\right), \]

and the inverse relations are
\[ x = r \cos \phi, \quad y = r \sin \phi. \]

**Figure 3-16:** Interrelationships between Cartesian coordinates \((x, y, z)\) and cylindrical coordinates \((r, \phi, z)\).
Coordinate Transformations: Unit Vectors

\[ \hat{r} \cdot \hat{x} = \cos \phi, \quad \hat{r} \cdot \hat{y} = \sin \phi, \]
\[ \dot{\phi} \cdot \hat{x} = -\sin \phi, \quad \dot{\phi} \cdot \hat{y} = \cos \phi. \]
\[ \hat{r} = \hat{x} \cos \phi + \hat{y} \sin \phi. \]
\[ \dot{\phi} = -\hat{x} \sin \phi + \hat{y} \cos \phi. \]
\[ \hat{x} = \hat{r} \cos \phi - \dot{\phi} \sin \phi, \]
\[ \hat{y} = \hat{r} \sin \phi + \dot{\phi} \cos \phi. \]
<table>
<thead>
<tr>
<th>Transformation</th>
<th>Coordinate Variables</th>
<th>Unit Vectors</th>
<th>Vector Components</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cartesian to cylindrical</strong></td>
<td>( r = \sqrt{x^2 + y^2} )  \  ( \phi = \tan^{-1}(y/x) )  \  ( z = z )</td>
<td>( \hat{r} = \hat{x} \cos \phi + \hat{y} \sin \phi )  \  ( \hat{\phi} = -\hat{x} \sin \phi + \hat{y} \cos \phi )  \  ( \hat{z} = \hat{z} )</td>
<td>( A_r = A_x \cos \phi + A_y \sin \phi )  \  ( A_\phi = -A_x \sin \phi + A_y \cos \phi )  \  ( A_z = A_z )</td>
</tr>
<tr>
<td><strong>Cylindrical to Cartesian</strong></td>
<td>( x = r \cos \phi )  \  ( y = r \sin \phi )  \  ( z = z )</td>
<td>( \hat{x} = \hat{r} \cos \phi - \hat{\phi} \sin \phi )  \  ( \hat{y} = \hat{r} \sin \phi + \hat{\phi} \cos \phi )  \  ( \hat{z} = \hat{z} )</td>
<td>( A_x = A_r \cos \phi - A_\phi \sin \phi )  \  ( A_y = A_r \sin \phi + A_\phi \cos \phi )  \  ( A_z = A_z )</td>
</tr>
<tr>
<td><strong>Cartesian to spherical</strong></td>
<td>( R = \sqrt{x^2 + y^2 + z^2} )  \  ( \theta = \tan^{-1}[\sqrt{x^2 + y^2}/z] )  \  ( \phi = \tan^{-1}(y/x) )</td>
<td>( \hat{R} = \hat{x} \sin \theta \cos \phi )  \  ( + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta )  \  ( \hat{\theta} = \hat{x} \cos \theta \cos \phi )  \  ( + \hat{y} \cos \theta \sin \phi - \hat{z} \sin \theta )  \  ( \hat{\phi} = -\hat{x} \sin \phi + \hat{y} \cos \phi )</td>
<td>( A_R = A_x \sin \theta \cos \phi )  \  ( + A_y \sin \theta \sin \phi + A_z \cos \theta )  \  ( A_\theta = A_x \cos \theta \cos \phi )  \  ( + A_y \cos \theta \sin \phi - A_z \sin \theta )  \  ( A_\phi = -A_x \sin \phi + A_y \cos \phi )</td>
</tr>
<tr>
<td><strong>Spherical to Cartesian</strong></td>
<td>( x = R \sin \theta \cos \phi )  \  ( y = R \sin \theta \sin \phi )  \  ( z = R \cos \theta )</td>
<td>( \hat{x} = \hat{R} \sin \theta \cos \phi )  \  ( + \hat{\theta} \cos \theta \cos \phi - \hat{\phi} \sin \phi )  \  ( \hat{y} = \hat{R} \sin \theta \sin \phi )  \  ( + \hat{\theta} \cos \theta \sin \phi + \hat{\phi} \cos \phi )  \  ( \hat{z} = \hat{R} \cos \theta - \hat{\theta} \sin \theta )</td>
<td>( A_x = A_R \sin \theta \cos \phi )  \  ( + A_\theta \cos \theta \cos \phi - A_\phi \sin \phi )  \  ( A_y = A_R \sin \theta \sin \phi )  \  ( + A_\theta \cos \theta \sin \phi + A_\phi \cos \phi )  \  ( A_z = A_R \cos \theta - A_\theta \sin \theta )</td>
</tr>
<tr>
<td><strong>Cylindrical to spherical</strong></td>
<td>( R = \sqrt{r^2 + z^2} )  \  ( \theta = \tan^{-1}(r/z) )  \  ( \phi = \phi )</td>
<td>( \hat{R} = \hat{r} \sin \theta + \hat{z} \cos \theta )  \  ( \hat{\theta} = \hat{r} \cos \theta - \hat{z} \sin \theta )  \  ( \hat{\phi} = \hat{\phi} )</td>
<td>( A_R = A_r \sin \theta + A_z \cos \theta )  \  ( A_\theta = A_r \cos \theta - A_z \sin \theta )  \  ( A_\phi = A_\phi )</td>
</tr>
<tr>
<td><strong>Spherical to cylindrical</strong></td>
<td>( r = R \sin \theta )  \  ( \phi = \phi )  \  ( z = R \cos \theta )</td>
<td>( \hat{r} = \hat{R} \sin \theta + \hat{\theta} \cos \theta )  \  ( \hat{\phi} = \hat{\phi} )  \  ( \hat{z} = \hat{R} \cos \theta - \hat{\theta} \sin \theta )</td>
<td>( A_r = A_R \sin \theta + A_\theta \cos \theta )  \  ( A_\phi = A_\phi )  \  ( A_z = A_R \cos \theta - A_\theta \sin \theta )</td>
</tr>
</tbody>
</table>
Example 3-7: Cartesian to Cylindrical Transformations

Given point \( P_1 = (3, -4, 3) \) and vector \( \mathbf{A} = \hat{x}2 - \hat{y}3 + \hat{z}4 \), defined in Cartesian coordinates, express \( P_1 \) and \( \mathbf{A} \) in cylindrical coordinates and evaluate \( \mathbf{A} \) at \( P_1 \).

**Solution:** For point \( P_1 \), \( x = 3 \), \( y = -4 \), and \( z = 3 \). Using Eq. (3.51), we have

\[
r = \sqrt{x^2 + y^2} = 5, \quad \phi = \tan^{-1} \frac{y}{x} = -53.1^\circ = 306.9^\circ,
\]
and \( z \) remains unchanged. Hence, \( P_1 = (5, 306.9^\circ, 3) \) in cylindrical coordinates.

The cylindrical components of vector \( \mathbf{A} = \hat{r}A_r + \hat{\phi}A_\phi + \hat{z}A_z \) can be determined by applying Eqs. (3.58a) and (3.58b):

\[
A_r = A_x \cos \phi + A_y \sin \phi = 2 \cos \phi - 3 \sin \phi,
\]
\[
A_\phi = -A_x \sin \phi + A_y \cos \phi = -2 \sin \phi - 3 \cos \phi,
\]
\[
A_z = 4.
\]
Hence,

\[
\mathbf{A} = \hat{r}(2 \cos \phi - 3 \sin \phi) - \hat{\phi}(2 \sin \phi + 3 \cos \phi) + \hat{z}4.
\]

At point \( P \), \( \phi = 306.9^\circ \), which gives

\[
\mathbf{A} = \hat{r}3.60 - \hat{\phi}0.20 + \hat{z}4.
\]
Example 3-8: Cartesian to Spherical Transformation

Express vector $\mathbf{A} = \hat{x}(x + y) + \hat{y}(y - x) + \hat{z}z$ in spherical coordinates.

**Solution:** Using the transformation relation for $A_R$ given in Table 3-2, we have

$$A_R = A_x \sin \theta \cos \phi + A_y \sin \theta \sin \phi + A_z \cos \theta$$
$$= (x + y) \sin \theta \cos \phi + (y - x) \sin \theta \sin \phi + z \cos \theta.$$

Using the expressions for $x$, $y$, and $z$ given by Eq. (3.61c), we have

$$A_R = (R \sin \theta \cos \phi + R \sin \theta \sin \phi) \sin \theta \cos \phi$$
$$+ (R \sin \theta \sin \phi - R \sin \theta \cos \phi) \sin \theta \sin \phi + R \cos^2 \theta$$
$$= R \sin^2 \theta (\cos^2 \phi + \sin^2 \phi) + R \cos^2 \theta$$
$$= R \sin^2 \theta + R \cos^2 \theta = R.$$

Similarly,

$$A_\theta = (x + y) \cos \theta \cos \phi + (y - x) \cos \theta \sin \phi - z \sin \theta,$$
$$A_\phi = -(x + y) \sin \phi + (y - x) \cos \phi,$$

and following the procedure used with $A_R$, we obtain

$$A_\theta = 0,$$
$$A_\phi = -R \sin \theta.$$

Hence,

$$\mathbf{A} = \mathbf{\hat{R}} A_R + \mathbf{\hat{\theta}} A_\theta + \mathbf{\hat{\phi}} A_\phi = \mathbf{\hat{R}} R - \mathbf{\hat{\phi}} R \sin \theta.$$
Distance Between 2 Points

\[ d = |\mathbf{R}_{12}| \]
\[ = \left[ (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \right]^{1/2}. \quad (3.66) \]

\[ d = \left[ (r_2 \cos \phi_2 - r_1 \cos \phi_1)^2 + (r_2 \sin \phi_2 - r_1 \sin \phi_1)^2 + (z_2 - z_1)^2 \right]^{1/2} \]
\[ = \left[ r_2^2 + r_1^2 - 2r_1 r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2 \right]^{1/2} \]
\[ \text{(cylindrical).} \quad (3.67) \]

\[ d = \left\{ R_2^2 + R_1^2 - 2R_1 R_2 \cos \theta_2 \cos \theta_1 + \sin \theta_1 \sin \theta_2 \cos(\phi_2 - \phi_1) \right\}^{1/2} \]
\[ \text{(spherical).} \quad (3.68) \]
In vector calculus, we use three fundamental “operators” to describe the spatial variation of scalars and vectors (fields)

One operates on scalar fields, the other two operate on vector fields.

GRADIENT (scalar)

DIVERGENCE (vector)

CURL (or ROTATION) (vector)
An example of a "scalar field" is one where temperature depends upon position in 3D space.

For example, on a weather map of US, one often sees "isolines" of (nast) temperature - like going from Lake Tahoe to Santa Cruz, temp lower in Tahoe than SC.

\[ T_1 < T_2 < T_3 \]

So for example, the differential change in $T$ across with respect to 3D coordinates (say, $x, y, z$) can be described as a partial derivative of $T$ with respect to the 3 coordinate axes.
Gradient of a Scalar Field

Thus, if the scalar (i.e., $T$) varies with position differently in different directions, we can speak of the change in these directions:

\[ dT = T_2 - T_1 = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz \]

In the box shown, the differential length vector from $P_1(x_1, y_1, z_1)$ to $P_2(x_1 + dx, y_1 + dy, z_1 + dz)$ is given by:

\[ d\mathbf{l} = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k} \]

Since by definition, the differential length in a direction is:

\[ dx = \frac{\partial}{\partial x} d\mathbf{r} \]
\[ dy = \frac{\partial}{\partial y} d\mathbf{r} \]
\[ dz = \frac{\partial}{\partial z} d\mathbf{r} \]

then

\[ dT = \left[ \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz \right] d\mathbf{r} \]
\[ = \nabla T \cdot d\mathbf{r} \]

We call \[ \nabla T \] the gradient of $T$. 
Gradient of A Scalar Field

From differential calculus, the temperature difference between points \( P_1 \) and \( P_2 \), \( dT = T_2 - T_1 \), is

\[
dT = \frac{\partial T}{\partial x} \, dx + \frac{\partial T}{\partial y} \, dy + \frac{\partial T}{\partial z} \, dz. \tag{3.70}
\]

Because \( dx = \hat{x} \cdot dl \), \( dy = \hat{y} \cdot dl \), and \( dz = \hat{z} \cdot dl \), Eq. (3.70) can be rewritten as

\[
dT = \hat{x} \frac{\partial T}{\partial x} \cdot dl + \hat{y} \frac{\partial T}{\partial y} \cdot dl + \hat{z} \frac{\partial T}{\partial z} \cdot dl
\]

\[
= \left[ \frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z} \right] \cdot dl. \tag{3.71}
\]

\[
\nabla T = \text{grad } T = \hat{x} \frac{\partial T}{\partial x} + \hat{y} \frac{\partial T}{\partial y} + \hat{z} \frac{\partial T}{\partial z}. \tag{3.72}
\]

Equation (3.71) can then be expressed as

\[
dT = \nabla T \cdot dl. \tag{3.73}
\]

The symbol \( \nabla \) is called the \textit{del} or \textit{gradient operator} and is defined as

\[
\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \quad \text{(Cartesian).} \tag{3.74}
\]
Gradient (cont.)

With \( d\mathbf{l} = \hat{\mathbf{a}}_l dl \), where \( \hat{\mathbf{a}}_l \) is the unit vector of \( d\mathbf{l} \), the directional derivative of \( T \) along \( \hat{\mathbf{a}}_l \) is

\[
\frac{dT}{dl} = \nabla T \cdot \hat{\mathbf{a}}_l. \tag{3.75}
\]

We can find the difference \( (T_2 - T_1) \), where \( T_1 = T(x_1, y_1, z_1) \) and \( T_2 = T(x_2, y_2, z_2) \) are the values of \( T \) at points \( P_1 = (x_1, y_1, z_1) \) and \( P_2 = (x_2, y_2, z_2) \) not necessarily infinitesimally close to one another, by integrating both sides of Eq. (3.73). Thus,

\[
T_2 - T_1 = \int_{P_1}^{P_2} \nabla T \cdot d\mathbf{l}. \tag{3.76}
\]

\[\text{Example 3-9: Directional Derivative}\]

Find the directional derivative of \( T = x^2 + y^2 z \) along direction \( \hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2 \) and evaluate it at \((1, -1, 2)\).

**Solution:** First, we find the gradient of \( T \):

\[
\nabla T = \left( \frac{\partial}{\partial x} x^2 + \frac{\partial}{\partial y} y^2 z + \frac{\partial}{\partial z} \right) (x^2 + y^2 z) = \hat{\mathbf{x}}2x + \hat{\mathbf{y}}2yz + \hat{\mathbf{z}}y^2.
\]

We denote \( \mathbf{l} \) as the given direction,

\[\mathbf{l} = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2.\]

Its unit vector is

\[
\hat{\mathbf{a}}_l = \frac{\mathbf{l}}{||\mathbf{l}||} = \frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2}{\sqrt{2^2 + 3^2 + 2^2}} = \frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2}{\sqrt{17}}.
\]

Application of Eq. (3.75) gives

\[
\frac{dT}{dl} = \nabla T \cdot \hat{\mathbf{a}}_l = \left( \hat{\mathbf{x}}2x + \hat{\mathbf{y}}2yz + \hat{\mathbf{z}}y^2 \right) \cdot \left( \hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2 \right) \sqrt{17} = \frac{4x + 6yz - 2y^2}{\sqrt{17}}.
\]

At \((1, -1, 2)\),

\[
\left. \frac{dT}{dl} \right|_{(1,-1,2)} = \frac{4 - 12 - 2}{\sqrt{17}} = -\frac{10}{\sqrt{17}}.
\]
Let's look at the "gradient" operator in context of what you have learned before.

Example: Electric potential

Remember, the electric field $\vec{E}$ is the force per unit charge under $\vec{E}$ field as shown—in $y$ direction.

If we move a charge $q$ in the direction, we need to push against the $\vec{E}$ field force.

Thus $\vec{F}_{\text{external}} = -q\vec{E}$—electric force. $\parallel \vec{F}_{\text{ext}} + \vec{F}_e = 0$ $\vec{F}_e = q\vec{E}$

Energy to do that is:

$dw = \vec{F}_{\text{ext}} \cdot dl$ (ie energy (work) = force moved.

Note that dot product tells us no work if we push perpendicular to path.

$dw = -q\vec{E} \cdot dl$ since $\vec{F}_e = q\vec{E}$ def. of electric field.

In this case $dl = dy$.

We define the electric potential as the work/unit charge to move a charge.

$\therefore \frac{dV}{dl} = -E \cdot dl$ relates potential to electric field.

Thus since we can write for any scalar field $\frac{dV}{dl} = \nabla V \cdot dl$.
Then \[ \nabla V = -E \quad \text{or} \quad E = -\nabla V \] the electric field is the negative of the gradient of the electric potential.

If we plot equipotential lines, \( E \) is in the negative direction of the max. rate of change of electric potential (i.e., change of \( V \) with respect to position). This will prove to be extremely useful since we will show how to calculate \( V \) for any distribution of electric charge and it is easier to calculate a scalar field than a vector one. (With a vector field, you basically have \( \mathbf{3} \) scalar calls.)

If these \( V_i \) are equipotential and \( \Delta V \) apart, then \( E \) direction as shown opp max rate of change of \( V \).
then using
\[ \nabla T = x \frac{\partial T}{\partial x} + y \frac{\partial T}{\partial y} + z \frac{\partial T}{\partial z} \]

and an expression for \( \frac{\partial T}{\partial x} = \frac{\partial T}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial T}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{\partial T}{\partial \theta} \frac{\partial \theta}{\partial x} + v \)

and for \( \frac{\partial T}{\partial y} \).

and an und. transforms for \( x, y \) we eventually get (exercise for student)
\[ \nabla \Phi = \hat{r} \frac{\partial \Phi}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial \Phi}{\partial \theta} + \hat{\phi} \frac{2}{r \sin \theta} \frac{\partial \Phi}{\partial \phi} \]

going thru the same exercise from \( x, y, z \) to \( r, \theta, \phi \)

in sph coords we get
\[ \nabla T = \hat{r} \frac{\partial T}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial T}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial T}{\partial \phi} \]

so our gradient operator becomes:
Rect.
\[ \nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \]

Cyl.
\[ \nabla = \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{z} \frac{\partial}{\partial z} \]

Sph.
\[ \nabla = \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \]
CD Module 3.2 Gradient

Select a scalar function \( f(x, y, z) \), evaluate its gradient, and display both in an appropriate 2-D plane.

\[
\nabla (\cos(x-y)) = \hat{x} \frac{d}{dx} (\cos(x-y)) + \hat{y} \frac{d}{dy} (\cos(x-y)) + \hat{z} \frac{d}{dz} (\cos(x-y))
\]

\[
= -\hat{x} \sin(x-y) + \hat{y} \sin(x-y)
\]

\[
= -\sin(x-y)(\hat{x} - \hat{y})
\]
Divergence of a Vector Field

At a surface boundary, *flux density* is defined as the amount of outward flux crossing a unit surface $ds$:

$$\text{Flux density of } \mathbf{E} = \frac{\mathbf{E} \cdot ds}{|ds|} = \frac{\mathbf{E} \cdot \mathbf{n} \, ds}{ds} = \mathbf{E} \cdot \mathbf{n}, \quad (3.85)$$

where $\mathbf{n}$ is the normal to $ds$. The *total flux* outwardly crossing a closed surface $S$, such as the enclosed surface of the imaginary sphere outlined in Fig. 3-20, is

$$\text{Total flux} = \oint_S \mathbf{E} \cdot ds. \quad (3.86)$$

$$\text{div} \mathbf{E} \triangleq \lim_{\Delta V \to 0} \frac{\int_S \mathbf{E} \cdot ds}{\Delta V}, \quad (3.95)$$

where $S$ encloses the elemental volume $\Delta V$. Instead of denoting the divergence of $\mathbf{E}$ by $\text{div} \mathbf{E}$, it is common practice to denote it as $\nabla \cdot \mathbf{E}$. That is,

$$\nabla \cdot \mathbf{E} = \text{div} \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \quad (3.96)$$

for a vector $\mathbf{E}$ in Cartesian coordinates.

*From the definition of the divergence of $\mathbf{E}$ given by Eq. (3.95), field $\mathbf{E}$ has positive divergence if the net flux out of surface $S$ is positive, which may be “viewed” as if volume $\Delta V$ contains a *source* of field lines. If the divergence is negative, $\Delta V$ may be viewed as containing a *sink* of field lines because the net flux is into $\Delta V$. For a uniform field $\mathbf{E}$, the same amount of flux enters $\Delta V$ as leaves it; hence, its divergence is zero and the field is said to be *divergenceless*. 
Divergence Theorem

$$\int_{V} \nabla \cdot \mathbf{E} \, dV = \oint_{S} \mathbf{E} \cdot d\mathbf{s} \quad \text{(divergence theorem).} \tag{3.98}$$

Useful tool for converting integration over a volume to one over the surface enclosing that volume, and vice versa.
Example 3-11: Calculating the Divergence

Determine the divergence of each of the following vector fields and then evaluate them at the indicated points:

(a) \( \mathbf{E} = \hat{x} 3x^2 + \hat{y} 2z + \hat{z} x^2 z \) at \((2, -2, 0)\);

(b) \( \mathbf{E} = \hat{R}(a^3 \cos \theta / R^2) - \hat{\theta}(a^3 \sin \theta / R^2) \) at \((a/2, 0, \pi)\).

Solution:

(a) \[ \nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \]
\[ = \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(x^2z) \]
\[ = 6x + 0 + x^2 \]
\[ = x^2 + 6x. \]

At \((2, -2, 0)\), \( \nabla \cdot \mathbf{E}\bigg|_{(2, -2, 0)} = 16. \)

(b) From the expression given on the inside of the back cover of the book for the divergence of a vector in spherical coordinates, it follows that

\[
\nabla \cdot \mathbf{E} = \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 E_R) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} (E_{\theta} \sin \theta) \\
+ \frac{1}{R \sin \theta} \frac{\partial E_{\phi}}{\partial \phi} \\
= \frac{1}{R^2} \frac{\partial}{\partial R} (a^3 \cos \theta) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} \left(-\frac{a^3 \sin^2 \theta}{R^2}\right) \\
= 0 - \frac{2a^3 \cos \theta}{R^3} \\
= -\frac{2a^3 \cos \theta}{R^3}. 
\]

At \( R = a/2 \) and \( \theta = 0 \), \( \nabla \cdot \mathbf{E}\bigg|_{(a/2, 0, \pi)} = -16. \)
CD Module 3.3 Divergence
Select a vector function \( \mathbf{v}(x, y, z) \), evaluate its divergence, and display both in an appropriate 2-D plane.
Curl of a Vector Field

\[ \text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l}. \]

\[ \nabla \times \mathbf{B} = \text{curl} \mathbf{B} \]
\[ = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[ \hat{n} \oint_C \mathbf{B} \cdot d\mathbf{l} \right]_{\text{max}}. \quad (3.103) \]

Thus, curl \( \mathbf{B} \) is the circulation of \( \mathbf{B} \) per unit area, with the area \( \Delta s \) of the contour \( C \) being oriented such that the circulation is maximum.

Figure 3-22: Circulation is zero for the uniform field in (a), but it is not zero for the azimuthal field in (b).
Stokes’s Theorem

Stokes’s theorem converts the surface integral of the curl of a vector over an open surface $S$ into a line integral of the vector along the contour $C$ bounding the surface $S$.

For the geometry shown in Fig. 3-23, Stokes’s theorem states

$$
\int_{S} (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_{C} \mathbf{B} \cdot d\mathbf{l} \quad \text{(Stokes’s theorem)},
$$

(3.107)

**Figure 3-23:** The direction of the unit vector $\mathbf{n}$ is along the thumb when the other four fingers of the right hand follow $d\mathbf{l}$. 
CD Module 3.4 Curl  Select a vector \( \mathbf{v}(x, y) \), evaluate its curl, and display both in the \( x-y \) plane.

Module 3.4 Curl

Input

Select a function: \( \mathbf{v}(x, y) = x\hat{i}\sin(\pi y) + y\hat{j}\sin(\pi x) \)

Function: \( \mathbf{v}(x, y) = \hat{x}\sin(\pi y) + \hat{y}\sin(\pi x) \)

\( \nabla \times \mathbf{v}(x, y) = \hat{x}\sin(\pi y) + \hat{y}\sin(\pi x) \) is

\( \nabla \times \mathbf{v} = \hat{z} (\pi \cos(\pi x) - \pi \cos(\pi y)) \)

\( \nabla \times \mathbf{v} = \hat{z} (\pi \cos(\pi x) + \pi \cos(\pi y)) \)

\( \nabla \times \mathbf{v} = \hat{z} f(x, y) \)

\( f(x, y) = \frac{d}{dx} (\sin(\pi x)) - \frac{d}{dy} (\sin(\pi y)) \)

\( = \pi \cos(\pi x) - \pi \cos(\pi y) \)

Graphics Created with Wolfram Mathematica®
Laplacian Operator

Laplacian of a Scalar Field

\[ \nabla^2 V = \nabla \cdot (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}. \quad (3.110) \]

Laplacian of a Vector Field

\[ \nabla^2 \mathbf{E} = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \mathbf{E} \]
\[ = \hat{x} \nabla^2 E_x + \hat{y} \nabla^2 E_y + \hat{z} \nabla^2 E_z \]

Useful Relation

\[ \nabla^2 \mathbf{E} = \nabla (\nabla \cdot \mathbf{E}) - \nabla \times (\nabla \times \mathbf{E}). \quad (3.113) \]
How does a CT scanner generate a 3-D image?
For each anatomical slice, the CT scanner generates on the order of $7 \times 10^5$ measurements (1,000 angular orientations x 700 detector channels).

Use of vector calculus allows the extraction of the 2-D image of a slice.

Combining multiple slices generates a 3-D scan.
Chapter 3 Relationships

Distance Between Two Points
\[ d = \left[ (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \right]^{1/2} \]
\[ d = \left[ r_2^2 + r_1^2 - 2r_1r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2 \right]^{1/2} \]
\[ d = \left[ R_2^2 + R_1^2 - 2R_1 R_2 \cos(\theta_2 \cos \theta_1 + \sin \theta_1 \sin \phi_2 \cos(\phi_2 - \phi_1)) \right]^{1/2} \]

Coordinate Systems Table 3-1
Coordinate Transformations Table 3-2

Vector Products
\[ \mathbf{A} \cdot \mathbf{B} = AB \cos \theta_{AB} \]
\[ \mathbf{A} \times \mathbf{B} = \mathbf{n} AB \sin \theta_{AB} \]
\[ \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \]
\[ \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \]

Divergence Theorem
\[ \int_{V} \nabla \cdot \mathbf{E} \, dV = \oint_{S} \mathbf{E} \cdot ds \]

Vector Operators
\[ \nabla T = \hat{x} \frac{\partial T}{\partial x} + \hat{y} \frac{\partial T}{\partial y} + \hat{z} \frac{\partial T}{\partial z} \]
\[ \nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \]
\[ \nabla \times \mathbf{B} = \hat{x} \left( \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) + \hat{y} \left( \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) + \hat{z} \left( \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \]
\[ \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \]
(see back cover for cylindrical and spherical coordinates)

Stokes’s Theorem
\[ \int_{S} (\nabla \times \mathbf{B}) \cdot ds = \oint_{C} \mathbf{B} \cdot dl \]