Reading: Chapter 3. entire chapter.
review of vector algebra and calculus
For next week: start Chapter 4. 4.1-4.4

Homework #3, due 2/7/13: Chapter 3, problems,
3.5 (b,c,d,f,g), 3.27, 3.42, 3.47, 3.51, 3.53

For laboratory: read lab.2 introduction before lab.

Lecture 7
Gradient of A Scalar Field

From differential calculus, the temperature difference between points $P_1$ and $P_2$, $dT = T_2 - T_1$, is

$$dT = \frac{\partial T}{\partial x} \, dx + \frac{\partial T}{\partial y} \, dy + \frac{\partial T}{\partial z} \, dz. \quad (3.70)$$

Because $dx = \hat{x} \cdot d\mathbf{l}$, $dy = \hat{y} \cdot d\mathbf{l}$, and $dz = \hat{z} \cdot d\mathbf{l}$, Eq. (3.70) can be rewritten as

$$dT = \hat{x} \frac{\partial T}{\partial x} \cdot d\mathbf{l} + \hat{y} \frac{\partial T}{\partial y} \cdot d\mathbf{l} + \hat{z} \frac{\partial T}{\partial z} \cdot d\mathbf{l}$$

$$= \left[ \frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z} \right] \cdot d\mathbf{l}. \quad (3.71)$$

Equation (3.71) can then be expressed as

$$dT = \nabla T \cdot d\mathbf{l}. \quad (3.73)$$

The symbol $\nabla$ is called the **del** or **gradient operator** and is defined as

$$\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \quad \text{(Cartesian).} \quad (3.74)$$
Gradient (cont.)

With $dl = \hat{a}_l dl$, where $\hat{a}_l$ is the unit vector of $dl$, the directional derivative of $T$ along $\hat{a}_l$ is

$$\frac{dT}{dl} = \nabla T \cdot \hat{a}_l. \quad (3.75)$$

We can find the difference $(T_2 - T_1)$, where $T_1 = T(x_1, y_1, z_1)$ and $T_2 = T(x_2, y_2, z_2)$ are the values of $T$ at points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ not necessarily infinitesimally close to one another, by integrating both sides of Eq. (3.73). Thus,

$$T_2 - T_1 = \int_{P_1}^{P_2} \nabla T \cdot dl. \quad (3.76)$$

Example 3-9: Directional Derivative

Find the directional derivative of $T = x^2 + y^2 z$ along direction $\hat{x}2 + \hat{y}3 - \hat{z}2$ and evaluate it at $(1, -1, 2)$.

Solution: First, we find the gradient of $T$:

$$\nabla T = \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) (x^2 + y^2 z) = \hat{x}2x + \hat{y}2yz + \hat{z}y^2.$$  

We denote $\mathbf{l}$ as the given direction, 

$$\mathbf{l} = \hat{x}2 + \hat{y}3 - \hat{z}2.$$  

Its unit vector is

$$\hat{a}_l = \frac{\mathbf{l}}{||\mathbf{l}||} = \frac{\hat{x}2 + \hat{y}3 - \hat{z}2}{\sqrt{2^2 + 3^2 + 2^2}} = \frac{\hat{x}2 + \hat{y}3 - \hat{z}2}{\sqrt{17}}.$$  

Application of Eq. (3.75) gives

$$\frac{dT}{dl} = \nabla T \cdot \hat{a}_l = (\hat{x}2x + \hat{y}2yz + \hat{z}y^2) \cdot \left( \frac{\hat{x}2 + \hat{y}3 - \hat{z}2}{\sqrt{17}} \right) = \frac{4x + 6yz - 2y^2}{\sqrt{17}}.$$  

At $(1, -1, 2)$,

$$\left. \frac{dT}{dl} \right|_{(1,-1,2)} = \frac{4 - 12 - 2}{\sqrt{17}} = -\frac{10}{\sqrt{17}}.$$
Exercise 3.12

\[ T(x) = T_1 + \frac{(T_2 - T_1)}{e^{-x} + 1} \]

temp. "gradient"  
land → sea

\[ T_1 \]  
\[ T_2 \]

sea 0 land x

\[ a) \text{ what is direction of } \nabla T? \]

\[ \nabla T = \hat{x} \frac{dT}{dx} + \hat{y} \frac{dT}{dy} + \hat{z} \frac{dT}{dz} \]

so in \( \hat{x} \) direction

\[ \therefore \nabla T = \hat{x} \frac{3}{2x} \left[ T_1 + \frac{(T_2 - T_1)}{e^{-x} + 1} \right] \]

\[ \nabla T = \hat{x} (T_2 - T_1) \left( \frac{e^{-x}}{(e^{-x} + 1)^2} \right) \]

max at \( x = 0 \)  
prove this
Convert from rect. to cyl. coordinates.

\[(x, y, z) \rightarrow (r, \theta, z)\]

Chain rule: \[
\frac{\Delta r}{\Delta x} = \frac{\Delta r}{\Delta \theta} \frac{\Delta \theta}{\Delta x} + \frac{\Delta r}{\Delta y} \frac{\Delta y}{\Delta x} + \frac{\Delta r}{\Delta z} \frac{\Delta z}{\Delta x}
\]

\[= 0\] (Orthog)

\[
\Gamma = \sqrt{x^2 + y^2} \quad \tan \phi = \frac{y}{x}
\]

\[
\frac{d\Gamma}{dx} = \frac{x}{\sqrt{x^2 + y^2}} = \tan \phi
\]

\[
= \frac{x}{x^2 + y^2}
\]

\[
\frac{d\theta}{dx} = \frac{d}{dx} \left( \arctan \left( \frac{y}{x} \right) \right)
\]

\[
\frac{d}{dx} \left( \arctan \left( \frac{y}{x} \right) \right) = \frac{1}{1 + \left( \frac{y}{x} \right)^2}
\]

\[
\frac{2y}{x^2 + y^2} = -\frac{y}{x^2 + y^2} = \frac{1}{\Gamma} \frac{1}{r} = \frac{1}{r} \sin \phi = \frac{d\phi}{dx}
\]

\[
\frac{d\Gamma}{dx} = \frac{\Delta r}{d\theta} \cos \phi + \frac{\Delta r}{dy} \left( -\frac{1}{r} \sin \phi \right)
\]
Current $\nabla T$ from rect. to cyl. coords.

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial r} \frac{dr}{dy} + \frac{\partial T}{\partial \phi} \frac{d\phi}{dy} + \frac{\partial T}{\partial z} \frac{dz}{dy} \tag{2}$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial \phi}{\partial y} = \frac{d}{dy} \left( \text{arctan} \left( \frac{y}{x} \right) \right) = \frac{x}{x^2+y^2} \cdot \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r} \cos \phi = \frac{dx}{dy} \left( \frac{1}{r} \cos \phi \right)$$

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial r} \sin \phi + \frac{\partial T}{\partial \phi} \left( \frac{1}{r} \cos \phi \right)$$

$$\nabla T = x \frac{\partial T}{\partial x} + y \frac{\partial T}{\partial y} + z \frac{\partial T}{\partial z}$$

wind. transform \[ \hat{x} = r \cos \phi - \phi \sin \phi \]

\[ \hat{y} = r \sin \phi + \phi \cos \phi \]

\[ \hat{z} = 0 \]

work out the algebra to get

$$\nabla T = \hat{r} \frac{\partial T}{\partial r} + \hat{\phi} \frac{1}{r} \frac{\partial T}{\partial \phi} + \hat{z} \frac{\partial T}{\partial z}$$
Divergence of a Vector Field

At a surface boundary, flux density is defined as the amount of outward flux crossing a unit surface $ds$:

$$\text{Flux density of } \mathbf{E} = \frac{\mathbf{E} \cdot ds}{|ds|} = \frac{\mathbf{E} \cdot \hat{n} \, ds}{ds} = \mathbf{E} \cdot \hat{n},$$  \hspace{1cm} (3.85)

where $\hat{n}$ is the normal to $ds$. The total flux outwardly crossing a closed surface $S$, such as the enclosed surface of the imaginary sphere outlined in Fig. 3-20, is

$$\text{Total flux} = \oint_S \mathbf{E} \cdot ds.$$  \hspace{1cm} (3.86)

From the definition of the divergence of $\mathbf{E}$ given by Eq. (3.95), field $\mathbf{E}$ has positive divergence if the net flux out of surface $S$ is positive, which may be “viewed” as if volume $\Delta V$ contains a source of field lines. If the divergence is negative, $\Delta V$ may be viewed as containing a sink of field lines because the net flux is into $\Delta V$. For a uniform field $\mathbf{E}$, the same amount of flux enters $\Delta V$ as leaves it; hence, its divergence is zero and the field is said to be divergenceless.
Divergence

\[ \vec{E} = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z \]

Flux through face 1 is:
\[ F_1 = \int \vec{E} \cdot \hat{n}_1 \, ds = \int (\hat{x}E_x + \hat{y}E_y + \hat{z}E_z) \cdot \left( -\hat{x} \right) dy \, dz \]

Since $\hat{x} \cdot \hat{y} = \hat{x} \cdot \hat{z} = 0$ and $\hat{x} \cdot \hat{x} = 1$

\[ F_1 = -E_x(1)dydz \quad \text{at center of face} \]

Flux through face 2 is: with $\hat{n}_2 = +\hat{x}$
\[ F_2 = +E_x(2)dydz \]

There is a variation $\Delta x$ between face 1 and 2,
\[ E_x(2) = E_x(1) + \frac{\partial E_x}{\partial x} \Delta x + \ldots + O(\Delta x^2) \]

\[ F_2 = \left[ E_x(1) + \frac{\partial E_x}{\partial x} \Delta x \right] dydz \]

and $F_1 + F_2 = \frac{\partial E_x}{\partial x} \Delta x \Delta y \Delta z$

Same thing for the other face pairs:
\[ F_3 + F_4 = \frac{\partial E_y}{\partial y} \Delta x \Delta y \Delta z \]
\[ F_5 + F_6 = \frac{\partial E_z}{\partial z} \Delta x \Delta y \Delta z \]

\[ \sum_{i=1}^{6} F_i = \int \vec{E} \cdot d\mathbf{s} = \left( \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) \Delta x \Delta y \Delta z \]
Lecture 7.

\[ \mathbf{F} \cdot d\mathbf{s} = \mathbf{\nabla} \cdot \mathbf{E} \Delta V \]

then we can extend to ANY volume

\[ \int_S \mathbf{F} \cdot d\mathbf{s} = \int_V \mathbf{\nabla} \cdot \mathbf{E} dV \]

— independent of coordinate system used —

as long as there is a SOURCE of FLUX within the closed surface, \( \mathbf{\nabla} \cdot \mathbf{E} \neq 0 \)

\[ \mathbf{\nabla} \cdot \mathbf{E} > 0 \] if source within volume
\[ \mathbf{\nabla} \cdot \mathbf{E} < 0 \] if sink within volume
DIVERGENCE
of vector fields

Positive

Negative
Divergence Theorem

\[ \int_{V} \nabla \cdot \mathbf{E} \, dV = \oint_{S} \mathbf{E} \cdot d\mathbf{s} \quad \text{(divergence theorem).} \]

(3.98)

Useful tool for converting integration over a volume to one over the surface enclosing that volume, and vice versa
Example 3-11: Calculating the Divergence

Determine the divergence of each of the following vector fields and then evaluate them at the indicated points:

(a) \( \mathbf{E} = \hat{x}3x^2 + \hat{y}2z + \hat{z}x^2z \) at \((2, -2, 0)\);

(b) \( \mathbf{E} = \hat{R}(a^3 \cos \theta / R^2) - \hat{\theta}(a^3 \sin \theta / R^2) \) at \((a/2, 0, \pi)\).

Solution:

(a) \( \nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \)

\[ = \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(x^2z) \]

\[ = 6x + 0 + x^2 \]

\[ = x^2 + 6x. \]

At \((2, -2, 0)\), \( \nabla \cdot \mathbf{E} \bigg|_{(2,-2,0)} = 16. \)

(b) From the expression given on the inside of the back cover of the book for the divergence of a vector in spherical coordinates, it follows that

\[ \nabla \cdot \mathbf{E} = \frac{1}{R^2} \frac{\partial}{\partial R} \left( R^2 E_R \right) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} \left( E_{\theta} \sin \theta \right) \]

\[ + \frac{1}{R \sin \theta} \frac{\partial}{\partial \phi} E_{\phi} \]

\[ = \frac{1}{R^2} \frac{\partial}{\partial R} \left( a^3 \cos \theta \right) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} \left( -a^3 \sin^2 \theta \right) \]

\[ = 0 - \frac{2a^3 \cos \theta}{R^3} \]

\[ = -\frac{2a^3 \cos \theta}{R^3}. \]

At \( R = a/2 \) and \( \theta = 0 \), \( \nabla \cdot \mathbf{E} \bigg|_{(a/2,0,\pi)} = -16. \)
CD Module 3.3 Divergence
Select a vector function $\mathbf{v}(x, y, z)$, evaluate its divergence, and display both in an appropriate 2-D plane.

Input
Select a function: $\mathbf{v}(x, y)$ as electric field due to 2 point charges
Function: $\mathbf{v}(x, y)$ is the electric field due to a positive charge at $(0, 0, 0)$ (point 1) and a negative one at $(0.5, 0.5, 0)$ (point 2)

The divergence of the electric field due to a positive charge at the origin and a negative one at $(0.5, 0.5, 0)$ ...

- \(\bigtriangledown\cdot \mathbf{v}\) has a source at the origin and a sink at $(0.5, 0.5, 0)$
- \(\bigtriangledown\cdot \mathbf{v}\) has a source at $(0.5, 0.5, 0)$ and a sink at the origin

Plot $\mathbf{v}$ and $\bigtriangledown\cdot \mathbf{v}$ in the:
- $x$-$y$ plane (arrows for $\mathbf{v}$, colors for $\bigtriangledown\cdot \mathbf{v}$)
- $x$-$y$ plane (field lines for $\mathbf{v}$, colors for $\bigtriangledown\cdot \mathbf{v}$)
- $x$-$z$ plane (arrows for $\mathbf{v}$, colors for $\bigtriangledown\cdot \mathbf{v}$)
- $x$-$z$ plane (field lines for $\mathbf{v}$, colors for $\bigtriangledown\cdot \mathbf{v}$)
- $y$-$z$ plane (arrows for $\mathbf{v}$, colors for $\bigtriangledown\cdot \mathbf{v}$)
- $y$-$z$ plane (field lines for $\mathbf{v}$, colors for $\bigtriangledown\cdot \mathbf{v}$)

$\bigtriangledown\cdot \mathbf{v} = \mathbf{v}(x, y, z) - \mathbf{v}(x-0.5, y-0.5, z-0.5)$
For example, consider the function \( \mathbf{F}(x, y) = (y, -x) \). We calculate values of the function at a set of points, such as

\[
\begin{align*}
\mathbf{F}(1, 0) &= (0, -1), \\
\mathbf{F}(0, 1) &= (1, 0), \\
\mathbf{F}(1, 1) &= (1, -1), \\
\mathbf{F}(1, 2) &= (2, -1).
\end{align*}
\]

By plotting each of these vectors anchored at the corresponding points, we begin to see some of the structure of the vector field.

If we continued plotting such vectors at many points, they would begin to overlap and look quite messy. Hence, we typically scale the arrows in vector field plots (usually without commenting on this fact). In the below example, we drew the vectors at only 40% their actual length. By plotting this field of arrows, we see that the vector field \( \mathbf{F}(x, y) = (y, -x) \) appears to rotate in a clockwise direction.
If we continued plotting such vectors at many points, they would begin to overlap and look quite messy. Hence, we typically scale the arrows in vector field plots (usually without commenting on this fact). In the below example, we drew the vectors at only 40% their actual length. By plotting this field of arrows, we see that the vector field $\mathbf{F}(x, y) = (y, -x)$ appears to rotate in a clockwise direction.
The vector field $\mathbf{F}(x, y, z) = (y/z, -x/z, 0)$ corresponds to a rotation in three dimensions, where the vector rotates around the $z$-axis. This vector field is similar to the two-dimensional rotation above. In this case, since we divided by $z$, the magnitude of the vector field decreases as $z$ increases.
Curl of a Vector Field

\[ \oint_C \mathbf{B} \cdot d\mathbf{l} = \text{circulation.} \]

A measure of "circulation" -

considers uniform \( \mathbf{B} \) field.

e.g. \( \mathbf{B} = \mathbf{\hat{x}} B_0 \)

\[ \oint_C \mathbf{B} \cdot d\mathbf{l} = 0 \]

\[ \int_a^c \mathbf{\hat{x}} B_0 \cdot \mathbf{\hat{x}} dx + \int_b^c \mathbf{\hat{y}} B_0 \cdot \mathbf{\hat{y}} dy + \int_{x B_0 \cdot \mathbf{\hat{x}}}^{x B_0 \cdot \mathbf{\hat{y}}} \]

\[ = 0 \text{ since } \mathbf{\hat{x}} \cdot \mathbf{\hat{y}} = 0 \]

\[ = B_0 \Delta x + 0 + (-B_0 \Delta x) + 0 \equiv 0 \]

circulation of a uniform \( \mathbf{B} \)-field does \( \not\equiv \)

\( \mathbf{\hat{n}} \) / righthand rule

\[ \nabla \times \mathbf{B} = \text{curl} \mathbf{B} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} [ \mathbf{n} \cdot (\mathbf{\hat{n}} \cdot d\mathbf{l}) ] \]

\( \mathbf{\Delta S} \) surface unpaired

by \( \mathbf{\hat{n}} \) - hand rule
Curl of a Vector Field

\[ \text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l}. \]

\[ \nabla \times \mathbf{B} = \text{curl } \mathbf{B} \]

\[ = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[ \hat{n} \oint_C \mathbf{B} \cdot d\mathbf{l} \right]_{\text{max}} \quad (3.103) \]

Thus, \( \text{curl } \mathbf{B} \) is the circulation of \( \mathbf{B} \) per unit area, with the area \( \Delta s \) of the contour \( C \) being oriented such that the circulation is maximum.

Figure 3-22: Circulation is zero for the uniform field in (a), but it is not zero for the azimuthal field in (b).
Magnetic Flux Density, $\mathbf{B}$, due to a current carrying wire (dc current = I)

$$\mathbf{B} = \frac{\hat{z} Q \mu_0 I}{2\pi r}$$

If we take a circular contour around the wire:

$$d\mathbf{l} = \hat{z} rd\theta$$

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \int_0^{2\pi} \hat{z} \frac{\mu_0 I}{2\pi r} \cdot \hat{z} r \, d\theta = \mu_0 I$$

But if we choose a contour in the $xy$ plane (in $\hat{x}$ direction, axis of wire) there is no $z$ component, so $\oint_C \mathbf{B} \cdot d\mathbf{l} = 0$

The circulation of $\mathbf{B}$ depends upon the contour and direction in which it is traversed.

direction of $\nabla \times \mathbf{B}$ is $\hat{n}$, the unit normal of $dS$ and right hand rule: right hand fingers curl along contours, thumb points along $\hat{n}$
Maxwell’s Equations

\[ \nabla \cdot D = \rho_v, \]
\[ \nabla \times E = -\frac{\partial B}{\partial t}, \]
\[ \nabla \cdot B = 0, \]
\[ \nabla \times H = J + \frac{\partial D}{\partial t}. \]

Under static conditions, none of the quantities appearing in Maxwell’s equations are functions of time (i.e., \( \partial / \partial t = 0 \)). This happens when all charges are permanently fixed in space, or, if they move, they do so at a steady rate so that \( \rho_v \) and \( J \) are constant in time. Under these circumstances, the time derivatives of \( B \) and \( D \) in Eqs. (4.1b) and (4.1d) vanish, and Maxwell’s equations reduce to

**Electrostatics**

\[ \nabla \cdot D = \rho_v, \quad (4.2a) \]
\[ \nabla \times E = 0. \quad (4.2b) \]

**Magnetostatics**

\[ \nabla \cdot B = 0, \quad (4.3a) \]
\[ \nabla \times H = J. \quad (4.3b) \]

Electric and magnetic fields become decoupled under static conditions.
CD Module 3.4 Curl Select a vector \( \mathbf{v}(x, y) \), evaluate its curl, and display both in the \( x-y \) plane.

**Input**
- Select a function: \( \mathbf{v}(x, y) = \hat{x}\sin(\pi y) + \hat{y}\sin(\pi x) \)
- Function: \( \mathbf{v}(x, y) = \hat{x}\sin(\pi y) + \hat{y}\sin(\pi x) \)
- \( \nabla \times \mathbf{v}(x, y) = \hat{x}\sin(\pi y) + \hat{y}\sin(\pi x) \)
  - \( \hat{z} \left( \pi\cos(\pi x) - \pi\cos(\pi y) \right) \)
  - \( \hat{z} \left( \pi\cos(\pi x) + \pi\cos(\pi y) \right) \)

- \( \mathbf{v}(x, y) \) (arrows) and \( f(x, y) \) (colors)
- \( \mathbf{v}(x, y) \) (lines) and \( f(x, y) \) (colors)

\[ \nabla \times \mathbf{v} = \hat{z} f(x, y) \]
\[ f(x, y) = \frac{d(\sin(\pi x))}{dx} - \frac{d(\sin(\pi y))}{dy} \]
\[ = \pi\cos(\pi x) - \pi\cos(\pi y) \]
Stokes’s Theorem

*Stokes’s theorem* converts the surface integral of the curl of a vector over an open surface $S$ into a line integral of the vector along the contour $C$ bounding the surface $S$.

For the geometry shown in Fig. 3-23, *Stokes’s theorem* states

$$
\int_S (\nabla \times \mathbf{B}) \cdot ds = \oint_C \mathbf{B} \cdot d\mathbf{l} \quad \text{(Stokes’s theorem)},
$$

(3.107)

**Figure 3-23:** The direction of the unit vector $\mathbf{n}$ is along the thumb when the other four fingers of the right hand follow $d\mathbf{l}$. 

Stokes Theorem

\[ \int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \oint_C \mathbf{B} \cdot d\mathbf{r} \]

where contour \( C \) bounds surface \( S \).

Example 3.12

Consider surface as shown

vector-field \( \mathbf{B} = \frac{\lambda}{r} \)

bounded by:

\[ r = 2 \]
\[ \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2} \]
\[ 0 \leq z \leq 3 \]
Stokes' Theorem

\[ \mathbf{B} = \frac{1}{2} \omega \times \mathbf{r} \rightarrow \text{choose cyl. corrd.} \]

\[ \therefore B_\theta = B_r = 0! \]

\[ \therefore \nabla \times \mathbf{B} = \nabla \left( \frac{1}{r^2} \frac{\partial}{\partial \theta} \right) \mathbf{B}_\theta - \frac{\partial}{\partial \phi} \frac{\mathbf{B}_r}{r} \]

\[ + \frac{\partial}{\partial \phi} \left( \frac{\mathbf{B}_\phi}{r} \right) \]

\[ + \frac{1}{r} \left( \frac{\partial}{\partial r} (r \mathbf{B}_r) - \frac{\partial}{\partial \theta} (r \mathbf{B}_\theta) \right) \]

\[ = \frac{1}{r} \left( \frac{\partial}{\partial \theta} \left( \frac{\mathbf{B}_r}{r} \right) \right) - \frac{\partial}{\partial \phi} \left( \frac{\mathbf{B}_r}{r} \right) \]

\[ - \mathbf{B}_r \frac{\sin \theta}{r^2} + \mathbf{B}_\phi \frac{\cos \theta}{r^2} \]

\[ \therefore \int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \int_0^{2\pi} \int_{\theta=0}^{\theta=\pi/3} \left[ -\mathbf{B}_r \frac{\sin \theta}{r^2} + \mathbf{B}_\phi \frac{\cos \theta}{r^2} \right] \hat{\mathbf{r}} \, d\theta \, d\phi \]

\[ \hat{\mathbf{n}} = \hat{\mathbf{r}} \]

\[ \mathbf{r} \cdot \hat{\mathbf{Q}} = 0 \]

\[ \int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \int_0^r \int_{\theta=0}^{\theta=\pi/3} \left( -\mathbf{B}_r \frac{\sin \theta}{r^2} \right) \, r \, dz \, d\theta \]

\[ = -\frac{3}{2r} = -\frac{3}{4} = \int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} \]
Stokes' Theorem

\[ \int_c \mathbf{B} \cdot d\mathbf{r} = \int_0^b \mathbf{B}_{ab} \cdot d\mathbf{e} + \int_b^c \mathbf{B}_{bc} \cdot d\mathbf{e} + \int_c^d \mathbf{B}_{cd} \cdot d\mathbf{e} + \int_d^a \mathbf{B}_{da} \cdot d\mathbf{e} \]

Note that \( \mathbf{B} = \mathbf{E} \) and \( \mathbf{E} \cdot d\mathbf{e} \) is a function of \( \mathbf{e} \),
so only \( d\mathbf{e} \) in \( z \) direction
\( \mathbf{E} \) to some only \( \mathbf{E} \cdot \mathbf{e} = 0 \)
all the rest are 0

\[ \therefore \int_c^b = 0 = \int_c^d \]

\[ \therefore \int_c^b \mathbf{B} \cdot d\mathbf{r} = \int_b^c \mathbf{B}_{bc} \cdot d\mathbf{e} + \int_c^d \mathbf{B}_{cd} \cdot d\mathbf{e} \]

\( \mathbf{B}_{bc} = \frac{3 \mathbf{w}_2}{r} = 0 \)  \( \mathbf{B}_{da} = \frac{3 \mathbf{w}_1}{2} = \frac{3}{4} \)

\[ \therefore \int_c^b \mathbf{B} \cdot d\mathbf{r} = \int_d^c (\frac{3}{4}) \cdot \mathbf{e}_z \cdot d\mathbf{z} = \int_0^{\frac{\pi}{4}} d\mathbf{z} = -\frac{3}{4} \]

Same as \( \int \mathbf{E} \times \mathbf{B} \cdot d\mathbf{s} \)
Laplacian of a Scalar Field

\[ \nabla^2 V = \nabla \cdot (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}. \]  \hspace{1cm} (3.110)

Laplacian of a Vector Field

\[ \nabla^2 \mathbf{E} = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \mathbf{E} \]
\[ = \hat{x} \nabla^2 E_x + \hat{y} \nabla^2 E_y + \hat{z} \nabla^2 E_z \]

Useful Relation

\[ \nabla^2 \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla \times (\nabla \times \mathbf{E}). \]  \hspace{1cm} (3.113)
Tech Brief 6: X-Ray Computed Tomography

How does a CT scanner generate a 3-D image?
Tech Brief 6: X-Ray Computed Tomography

- For each anatomical slice, the CT scanner generates on the order of $7 \times 10^5$ measurements (1,000 angular orientations x 700 detector channels)
- Use of vector calculus allows the extraction of the 2-D image of a slice
- Combining multiple slices generates a 3-D scan

Figure TF6-3: Basic elements of a CT scanner.
Image Reconstruction by Back Projection
Chapter 3 Relationships

Distance Between Two Points
\[ d = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2} \]
\[ d = [r_2^2 + r_1^2 - 2r_1r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2]^{1/2} \]
\[ d = \left\{ R_2^2 + R_1^2 - 2R_1 R_2 \left[ \cos \theta_2 \cos \theta_1 + \sin \theta_1 \sin \theta_2 \cos(\phi_2 - \phi_1) \right] \right\}^{1/2} \]

Vector Operators
\[ \nabla T = \hat{\mathbf{x}} \frac{\partial T}{\partial x} + \hat{\mathbf{y}} \frac{\partial T}{\partial y} + \hat{\mathbf{z}} \frac{\partial T}{\partial z} \]
\[ \nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \]
\[ \nabla \times \mathbf{B} = \hat{\mathbf{x}} \left( \frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y} \right) + \hat{\mathbf{y}} \left( \frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z} \right) + \hat{\mathbf{z}} \left( \frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} \right) \]
\[ \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \]
(see back cover for cylindrical and spherical coordinates)

Stokes’s Theorem
\[ \int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_C \mathbf{B} \cdot d\mathbf{l} \]
Chapter 4 Overview

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Objectives

Upon learning the material presented in this chapter, you should be able to:

1. Evaluate the electric field and electric potential due to any distribution of electric charges.
2. Apply Gauss’s law.
3. Calculate the resistance $R$ of any shaped object, given the electric field at every point in its volume.
4. Describe the operational principles of resistive and capacitive sensors.
5. Calculate the capacitance of two-conductor configurations.
Static vs. Dynamic

Static conditions: charges are stationary or moving, but if moving, they do so at a constant velocity.

Table 1-3: The three branches of electromagnetics.

<table>
<thead>
<tr>
<th>Branch</th>
<th>Condition</th>
<th>Field Quantities (Units)</th>
</tr>
</thead>
</table>
| Electrostatics          | Stationary charges $(\partial q/\partial t = 0)$ | Electric field intensity $E$ (V/m)  
Electric flux density $D$ (C/m$^2$)  
$D = \varepsilon E$ |
| Magnetostatics          | Steady currents $(\partial I/\partial t = 0)$ | Magnetic flux density $B$ (T)  
Magnetic field intensity $H$ (A/m)  
$B = \mu H$ |
| Dynamics (Time-varying fields) | Time-varying currents $(\partial I/\partial t \neq 0)$ | $E$, $D$, $B$, and $H$  
$(E, D)$ coupled to $(B, H)$ |

Under static conditions, electric and magnetic fields are independent, but under dynamic conditions, they become coupled.
Dynamical theory of the Electromagnetic field”, Philosophical Transactions of the Royal Society of London. 155. 459-512 (1865)
Maxwell’s Equations

\[ \nabla \cdot \mathbf{D} = \rho_v, \]
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \]
\[ \nabla \cdot \mathbf{B} = 0, \]
\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \]

Under static conditions, none of the quantities appearing in Maxwell’s equations are functions of time (i.e., \( \partial / \partial t = 0 \)). This happens when all charges are permanently fixed in space, or, if they move, they do so at a steady rate so that \( \rho_v \) and \( \mathbf{J} \) are constant in time. Under these circumstances, the time derivatives of \( \mathbf{B} \) and \( \mathbf{D} \) in Eqs. (4.1b) and (4.1d) vanish, and Maxwell’s equations reduce to

**Electrostatics**

\[ \nabla \cdot \mathbf{D} = \rho_v, \quad (4.2a) \]
\[ \nabla \times \mathbf{E} = 0. \quad (4.2b) \]

**Magnetostatics**

\[ \nabla \cdot \mathbf{B} = 0, \quad (4.3a) \]
\[ \nabla \times \mathbf{H} = \mathbf{J}. \quad (4.3b) \]

Electric and magnetic fields become decoupled under static conditions.
### Maxwell’s Equations

**Table 6-1:** Maxwell’s equations.

<table>
<thead>
<tr>
<th>Reference</th>
<th>Differential Form</th>
<th>Integral Form</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Gauss’s law</strong></td>
<td>( \nabla \cdot \mathbf{D} = \rho_v )</td>
<td>( \oint_S \mathbf{D} \cdot d\mathbf{s} = Q ) (6.1)</td>
</tr>
<tr>
<td><strong>Faraday’s law</strong></td>
<td>( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} )</td>
<td>( \oint_C \mathbf{E} \cdot d\mathbf{l} = -\oint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} ) (6.2)*</td>
</tr>
<tr>
<td><strong>Gauss’s law for magnetism</strong></td>
<td>( \nabla \cdot \mathbf{B} = 0 )</td>
<td>( \oint_S \mathbf{B} \cdot d\mathbf{s} = 0 ) (6.3)</td>
</tr>
<tr>
<td><strong>Ampère’s law</strong></td>
<td>( \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} )</td>
<td>( \oint_C \mathbf{H} \cdot d\mathbf{l} = \oint_S \left( \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s} ) (6.4)</td>
</tr>
</tbody>
</table>

*For a stationary surface \( S \).
On an atomic scale, charge is discrete. For this course in electromagnetics, we are generally talking about size scales several orders of magnitude greater in size than the atomic scale, so we generally ignore the discreteness of charge.

We speak of currents (the flow of charge) as continuing flowing charge and we speak of charge within an elemental volume as if it were uniformly distributed in that volume. That is, Charge density, $\rho = dq/dV$. 
Charge Distributions

Volume charge density:
\[ \rho_v = \lim_{\Delta V \to 0} \frac{\Delta q}{\Delta V} = \frac{dq}{dV} \quad \text{(C/m}^3\text{)} \]

Total Charge in a Volume
\[ Q = \int_V \rho_v \, dV \quad \text{(C)} \]

Surface and Line Charge Densities
\[ \rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds} \quad \text{(C/m}^2\text{)} \]
\[ \rho_\ell = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl} \quad \text{(C/m)} \]
Current Density

The amount of charge that crosses the tube’s cross-sectional surface $\Delta s'$ in time $\Delta t$ is therefore

$$\Delta q' = \rho_v \Delta V = \rho_v \Delta l \Delta s' = \rho_v u \Delta s' \Delta t. \quad (4.8)$$

For a surface with any orientation:

$$\Delta q = \rho_v u \cdot \Delta s \Delta t, \quad (4.9)$$

where $\Delta s = \hat{n} \Delta s$ and the corresponding total current flowing in the tube is

$$\Delta I = \frac{\Delta q}{\Delta t} = \rho_v u \cdot \Delta s = \mathbf{J} \cdot \Delta s, \quad (4.10)$$

where

$$\mathbf{J} = \rho_v u \quad (A/m^2) \quad (4.11)$$

$\mathbf{J}$ is called the current density.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{current_density_diagram.png}
\caption{Charges with velocity $\mathbf{u}$ moving through a cross section $\Delta s'$ in (a) and $\Delta s$ in (b).}
\end{figure}

When a current is due to the actual movement of electrically charged matter, it is called a convection current, and $\mathbf{J}$ is called a convection current density.
Convection vs. Conduction

When a current is due to the movement of charged particles relative to their host material, $J$ is called a conduction current density.

This movement of electrons from atom to atom constitutes a conduction current. The electrons that emerge from the wire are not necessarily the same electrons that entered the wire at the other end.

Conduction current, which is discussed in more detail in Section 4-6, obeys Ohm’s law, whereas convection current does not.
Coulomb’s Law

Electric field at point $P$ due to single charge

$$E = \frac{\hat{R}}{4\pi \varepsilon R^2} \quad \text{(V/m)}$$

Electric force on a test charge placed at $P$

$$F = q' E \quad \text{(N)}$$

Electric flux density $D$

$$D = \varepsilon E$$

$$\varepsilon = \varepsilon_r \varepsilon_0,$$

If $\varepsilon$ is independent of the magnitude of $E$, then the material is said to be linear because $D$ and $E$ are related linearly, and if it is independent of the direction of $E$, the material is said to be isotropic.

$$\varepsilon_0 = 8.85 \times 10^{-12} \approx (1/36\pi) \times 10^{-9} \quad \text{(F/m)}$$