EE 135, Winter 2013

Reading: Chapter 4. chapter 4.1-4.11

Homework #4, due 2/14: Chapter 4, problems, 4.10, 4.21, 4.24, 4.31, 4.46

For laboratory: read lab.2,3 introduction before lab.

Lecture 9
Current density

Consider a tube of charge with current density $j_V \text{ (m}^2/\text{m}^3)\) moving with a uniform velocity $\vec{v}$ along the axis.

Over a period of time $\Delta t$, the charge moves $\Delta x = \vec{v} \Delta t$.

Amount of charge that crosses a cross-sectional surface $\Delta S'$ in time $\Delta t$ is:

$$\Delta q' = j_V \Delta V = j_V \frac{\Delta V}{\Delta t} \Delta t = j_V \Delta V = j_V \Delta S' \cdot \Delta V$$

$$\Delta q' = j_V \Delta S' \cdot \Delta V$$

If surface $\Delta S'$ is not a plane, the amount of charge flowing is $\Delta q' = j_V \Delta \vec{S} \cdot \Delta \vec{V}$.
Current flow thru $\vec{A}$ is:

$$\Delta I = \frac{\Delta \Phi}{\Delta t} = \mathbf{J} \cdot \vec{A} \cdot \Delta \vec{S}.$$  Define $\mathbf{J} = \rho \mathbf{u}$ to be current density.

Then $\Delta I = \mathbf{J} \cdot \Delta \vec{S}$

$$|I| = \int_{\vec{S}} \mathbf{J} \cdot d\vec{s}$$

Current thru an arbitrary surface

2. types of current

1. convection current
   
   Actual movement of the charge.
   
   eg (wind driven charged cloud, electron beam in a CRT).

2. conduction current
   
   motion of the carriers (medium don't move from one end to another) only the carriers that get transmitted.

Note: because (1) & (2) generally unequal to diff.

physical mechanisms

Conduction current obeys $S$'s law,

Convection current does not!

We will deal with the latter thru our discussion

of real wires. More on conduction on Thursday
so now let's look at conductors in more detail (but not too much) property of conductor useful for E = \vec{E} - \text{That it is easy for current to flow.}

we call \( \vec{u_e} \) the drift vel of electrons in a conductor - it is related to the applied \( \vec{E} \) field,
\[
\vec{u_e} = -\mu_e \vec{E}
\]
(opp direction of \( \vec{E} \) since \( e^- \) in neg \( e^- \nabla \)

- mobility \( \mu_e \) - measure of how far a particle moves before it collides with an atom

In a metal, current flow due predominantly to elec -
But in a \( \text{semiconductors} \) you have elec + \( \text{h}^{+} \) holes (\(+\) charge carriers).
the drift vel of holes is \( \vec{u_h} = \mu_h \vec{E} \)

\( \mu_h \) - hole mobility

Current density in medium of charge dens \( \rho \) is
\[
\vec{J} = \rho \vec{V}
\]
\( \vec{V} \) - vel of the moving charge

with elec \& holes
\[
\vec{J} = \vec{J}_e + \vec{J}_h = \rho_e \vec{u}_e + \rho_h \vec{u}_h
\]
\( \rho_e \) - elec charge
\( \rho_h \) - hole charge

\[
\rho_e = N_e e\text{ elec charge}
\]
\( N_e \) - elec charge density
\( e \) - elec charge

\[
\rho_h = N_h e\text{ - hole charge}
\]
\( N_h \) - hole charge density

The conductivity \( \sigma \) of the material (which was defined before)
\[
\sigma = \frac{-\rho_e \mu_e + \rho_h \mu_h}{N_e e + N_h e} e = \text{semi}
\]
\[ \text{for a conductor (good) } N_{\text{free electrons}} \ll N_{\text{free holes}} \Rightarrow \text{electric equilibrium} \]

\[ \therefore \sigma = \text{Ne} \text{ne} \cdot E = \sigma \]

\[ \Rightarrow F = \sigma E \text{ A/m}^2 \]

which is another way of writing Ohm's law.

I leave it to you to show this result in \( V = IR \)

\[ \text{note: for a perfect conductor } \sigma \to \infty \]

\[ \therefore \frac{F}{\sigma} \to 0 = E \| \text{ie no electric field in a perfect conductor!} \]

\[ \text{for a perfect insulator (ie \# carriers) } \sigma \to 0 \]

\[ \therefore F \to 0 \text{ for insulator \# current flow!} \]

\( \sigma \approx 10^{-6} \text{ Henries/m for most metals.} \)

\[ \text{NOTE: we argued } E = 0 \text{ in conductor form before using Gauss law -} \]

\[ \text{Also - a perfect conductor is an equipotential!} \]

\[ \text{since } V_{Z1} = \sum_{p} \mathbf{E}_p \cdot d \mathbf{l} = - \int_{p} \mathbf{E} \cdot d\mathbf{l} = 0 \]

\[ \text{anywhere in the conductor!} \]

we will come back to this later! \]
Conduction Current

The conductivity of a material is a measure of how easily electrons can travel through the material under the influence of an externally applied electric field.

Conduction current density:

\[ J = \sigma E \quad \text{(A/m}^2\text{)} \quad \text{(Ohm’s law)}, \]

A perfect dielectric is a material with \( \sigma = 0 \). In contrast, a perfect conductor is a material with \( \sigma = \infty \). Some materials, called superconductors, exhibit such a behavior.

<table>
<thead>
<tr>
<th>Material</th>
<th>Conductivity, ( \sigma ) (S/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Conductors</strong></td>
<td></td>
</tr>
<tr>
<td>Silver</td>
<td>( 6.2 \times 10^7 )</td>
</tr>
<tr>
<td>Copper</td>
<td>( 5.8 \times 10^7 )</td>
</tr>
<tr>
<td>Gold</td>
<td>( 4.1 \times 10^7 )</td>
</tr>
<tr>
<td>Aluminum</td>
<td>( 3.5 \times 10^7 )</td>
</tr>
<tr>
<td>Iron</td>
<td>( 10^7 )</td>
</tr>
<tr>
<td>Mercury</td>
<td>( 10^6 )</td>
</tr>
<tr>
<td>Carbon</td>
<td>( 3 \times 10^4 )</td>
</tr>
<tr>
<td><strong>Semiconductors</strong></td>
<td></td>
</tr>
<tr>
<td>Pure germanium</td>
<td>( 2.2 )</td>
</tr>
<tr>
<td>Pure silicon</td>
<td>( 4.4 \times 10^{-4} )</td>
</tr>
<tr>
<td><strong>Insulators</strong></td>
<td></td>
</tr>
<tr>
<td>Glass</td>
<td>( 10^{-12} )</td>
</tr>
<tr>
<td>Paraffin</td>
<td>( 10^{-15} )</td>
</tr>
<tr>
<td>Mica</td>
<td>( 10^{-15} )</td>
</tr>
<tr>
<td>Fused quartz</td>
<td>( 10^{-17} )</td>
</tr>
</tbody>
</table>

Note how wide the range is, over 24 orders of magnitude.
Conductivity

\[
\sigma = -\rho_v \mu_e + \rho_h \mu_h \\
= (N_e \mu_e + N_h \mu_h)e \quad \text{(S/m)} \quad \text{(semiconductor)},
\]

(4.67a)

and its unit is siemens per meter (S/m). For a good conductor, \(N_h \mu_h \ll N_e \mu_e\), and Eq. (4.67a) reduces to

\[
\sigma = -\rho_v \mu_e = N_e \mu_e e \quad \text{(S/m)} \\
\quad \text{(conductor).} \quad \text{(4.67b)}
\]

In view of Eq. (4.66), in a perfect dielectric with \(\sigma = 0\), \(\mathbf{J} = 0\) regardless of \(\mathbf{E}\), and in a perfect conductor with \(\sigma = \infty\), \(\mathbf{E} = \mathbf{J}/\sigma = 0\) regardless of \(\mathbf{J}\).

That is,

\[
\mathbf{J} = \sigma \mathbf{E} \quad \text{(A/m}^2\text{)} \quad \text{(Ohm’s law)},
\]

- \(\rho_v = \text{volume charge density of electrons}\)
- \(\rho_h = \text{volume charge density of holes}\)
- \(\mu_e = \text{electron mobility}\)
- \(\mu_h = \text{hole mobility}\)
- \(N_e = \text{number of electrons per unit volume}\)
- \(N_h = \text{number of holes per unit volume}\)

Perfect dielectric: \(\mathbf{J} = 0\).

Perfect conductor: \(\mathbf{E} = 0\).
Example 4-8: Conduction Current in a Copper Wire

A 2-mm-diameter copper wire with conductivity of $5.8 \times 10^7$ S/m and electron mobility of 0.0032 (m$^2$/V·s) is subjected to an electric field of 20 (mV/m). Find (a) the volume charge density of the free electrons, (b) the current density, (c) the current flowing in the wire, (d) the electron drift velocity, and (e) the volume density of the free electrons.
Solution:

(a)

\[ \rho_{vc} = -\frac{\sigma}{\mu_e} = -\frac{5.8 \times 10^7}{0.0032} = -1.81 \times 10^{10} \text{ (C/m}^3\text{)} . \]

(b)

\[ J = \sigma E = 5.8 \times 10^7 \times 20 \times 10^{-3} = 1.16 \times 10^6 \text{ (A/m}^2\text{)} . \]

(c)

\[ I = JA \]

\[ = J \left( \frac{\pi d^2}{4} \right) = 1.16 \times 10^6 \left( \frac{\pi \times 4 \times 10^{-6}}{4} \right) = 3.64 \text{ A} . \]

(d)

\[ u_e = -\mu_e E = -0.0032 \times 20 \times 10^{-3} = -6.4 \times 10^{-5} \text{ m/s} . \]

The minus sign indicates that \( u_e \) is in the opposite direction of \( E \).

(e)

\[ N_e = -\frac{\rho_{vc}}{e} = \frac{1.81 \times 10^{10}}{1.6 \times 10^{-19}} = 1.13 \times 10^{29} \text{ electrons/m}^3 . \]
Resistance

Longitudinal Resistor

\[
V = V_1 - V_2 = -\int_{x_2}^{x_1} \mathbf{E} \cdot d\mathbf{l}
= -\int_{x_2}^{x_1} \mathbf{\hat{x}} E_x \cdot \mathbf{\hat{x}} \, dl = E_x l \quad (V).
\] (4.68)

Using Eq. (4.63), the current flowing through the cross section \(A\) at \(x_2\) is

\[
I = \int_{A} \mathbf{J} \cdot d\mathbf{s} = \int_{A} \sigma \mathbf{E} \cdot d\mathbf{s} = \sigma E_x A \quad (A).
\] (4.69)

From \(R = \frac{V}{I}\), the ratio of Eq. (4.68) to Eq. (4.69) gives

\[
R = \frac{l}{\sigma A} \quad (\Omega).
\] (4.70)

For any conductor:

\[
R = \frac{V}{I} = \frac{-\int_{l} \mathbf{E} \cdot d\mathbf{l}}{\int_{s} \mathbf{J} \cdot d\mathbf{s}} = \frac{-\int_{l} \mathbf{E} \cdot d\mathbf{l}}{\int_{s} \sigma \mathbf{E} \cdot d\mathbf{s}}.
\]
Example 4-9: Conductance of Coaxial Cable

The radii of the inner and outer conductors of a coaxial cable of length \( l \) are \( a \) and \( b \), respectively (Fig. 4-15). The insulation material has conductivity \( \sigma \). Obtain an expression for \( G' \), the conductance per unit length of the insulation layer.

**Solution:** Let \( I \) be the total current flowing radially (along \( \hat{r} \)) from the inner conductor to the outer conductor through the insulation material. At any radial distance \( r \) from the axis of the center conductor, the area through which the current flows is \( A = 2\pi rl \). Hence,

\[
\mathbf{J} = \hat{r} \frac{I}{A} = \hat{r} \frac{I}{2\pi rl},
\]

and from \( \mathbf{J} = \sigma \mathbf{E} \),

\[
\mathbf{E} = \hat{r} \frac{I}{2\pi \sigma rl}.
\]

In a resistor, the current flows from higher electric potential to lower potential. Hence, if \( \mathbf{J} \) is in the \( \hat{r} \)-direction, the inner conductor must be at a higher potential than the outer conductor. Accordingly, the voltage difference between the conductors is

\[
V_{ab} = -\int_{b}^{a} \mathbf{E} \cdot d\mathbf{l} = -\int_{b}^{a} \frac{I}{2\pi \sigma l} \hat{r} \cdot \hat{r} \frac{dr}{r}
\]

\[
= \frac{I}{2\pi \sigma l} \ln \left( \frac{b}{a} \right).
\]

The conductance per unit length is then

\[
G' = \frac{G}{l} = \frac{1}{Rl} = \frac{I}{V_{abl}} = \frac{2\pi \sigma}{\ln(b/a)} \quad \text{(S/m)}.
\]

\( G' = 0 \) if the insulating material is air or a perfect dielectric with zero conductivity.
Table 2-1: Transmission-line parameters $R'$, $L'$, $G'$, and $C'$ for three types of lines.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Coaxial</th>
<th>Two-Wire</th>
<th>Parallel-Plate</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R'$</td>
<td>$\frac{R_s}{2\pi} \left( \frac{1}{a} + \frac{1}{b} \right)$</td>
<td>$\frac{2R_s}{\pi d}$</td>
<td>$\frac{2R_s}{w}$</td>
<td>$\Omega/m$</td>
</tr>
<tr>
<td>$L'$</td>
<td>$\frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right)$</td>
<td>$\frac{\mu}{\pi} \ln \left[ \left(\frac{D}{d}\right) + \sqrt{\left(\frac{D}{d}\right)^2 - 1} \right]$</td>
<td>$\frac{\mu h}{w}$</td>
<td>$\text{H/m}$</td>
</tr>
<tr>
<td>$G'$</td>
<td>$\frac{2\pi \sigma}{\ln\left(\frac{b}{a}\right)}$</td>
<td>$\frac{\pi \sigma}{\ln \left[ \left(\frac{D}{d}\right) + \sqrt{\left(\frac{D}{d}\right)^2 - 1} \right]}$</td>
<td>$\frac{\sigma w}{h}$</td>
<td>$\text{S/m}$</td>
</tr>
<tr>
<td>$C'$</td>
<td>$\frac{2\pi \varepsilon}{\ln\left(\frac{b}{a}\right)}$</td>
<td>$\frac{\pi \varepsilon}{\ln \left[ \left(\frac{D}{d}\right) + \sqrt{\left(\frac{D}{d}\right)^2 - 1} \right]}$</td>
<td>$\frac{\varepsilon w}{h}$</td>
<td>$\text{F/m}$</td>
</tr>
</tbody>
</table>

Notes: (1) Refer to Fig. 2-4 for definitions of dimensions. (2) $\mu$, $\varepsilon$, and $\sigma$ pertain to the insulating material between the conductors. (3) $R_s = \sqrt{\pi f \mu_c / \sigma_c}$. (4) $\mu_c$ and $\sigma_c$ pertain to the conductors. (5) If $(D/d)^2 \gg 1$, then $\ln \left[ \left(\frac{D}{d}\right) + \sqrt{\left(\frac{D}{d}\right)^2 - 1} \right] \simeq \ln(2D/d)$.

The pertinent **constitutive parameters** apply to all three lines and consist of two groups: (1) $\mu_c$ and $\sigma_c$ are the magnetic permeability and electrical conductivity of the conductors, and (2) $\varepsilon$, $\mu$, and $\sigma$ are the electrical permittivity, magnetic permeability, and electrical conductivity of the insulation material separating them.
Joule's Law

\[ P_{ve} = \text{volume density of free electron} \]
\[ P_{vh} = \text{volume density of holes} \]

\[ \text{volume NOT potent} \]

\[ \dot{Q}_e = P_{ve} \dot{V} \quad \dot{Q}_h = P_{vh} \dot{V} \]

Forces on electrons = \[ \vec{F}_e = \dot{Q}_e \vec{E} = P_{ve} \dot{V} \vec{E} \]

For holes = \[ \vec{F}_h = P_{vh} \dot{V} \vec{E} \]

Work done by \[ \vec{E} \] on moving charge \[ Q_e \] a distance \[ \Delta x \] = \[ \vec{F}_e \Delta x = W_e \]

\[ \text{for holes, } W_h = F_h \Delta x \]

\[ \Delta P = \frac{\Delta W}{\Delta t} = P_{ve} \Delta \vec{E} + P_{vh} \frac{\Delta \vec{V}}{\Delta t} \]
\[ = P_{ve} \dot{\vec{E}} + P_{vh} \dot{\vec{V}} \]
\[ = (P_{ve} \dot{\vec{E}} \cdot \dot{\vec{E}} + P_{vh} \dot{\vec{E}} \cdot \dot{\vec{V}}) \Delta V \]
\[ = \dot{E} \cdot (P_{ve} \dot{\vec{E}} + P_{vh} \dot{\vec{V}}) \Delta V \]

\[ \Rightarrow \frac{\Delta P}{\Delta V} = \dot{E} \cdot \oint A \Delta V \]

\[ \Rightarrow P = \int E \cdot A \Delta V \]
Joule’s Law.

Since $\vec{J} = \sigma \vec{E}$ then

$$P = \int_v \vec{E} \cdot \vec{J} \, dv = \int_v \sigma \vec{E} \vec{I}^2 \, dv$$

from which $\vec{E} \rightarrow E_x \hat{x}$

$$\Delta V = AL$$

$$P = \int_v 6 \vec{E}^2 \, dv = 6E_x AL \frac{E_x}{I} = 1V = I^2 R$$

\[\begin{array}{c}
\text{K} \quad l \quad \text{A} \\
I \quad \hat{E} \\
V
\end{array}\]
Dielectric Materials

**Figure 4-16:** In the absence of an external electric field $E$, the center of the electron cloud is co-located with the center of the nucleus, but when a field is applied, the two centers are separated by a distance $d$.

**Figure 4-17:** A dielectric medium polarized by an external electric field $E$. 

(Equation images and description not shown in the natural text representation)
Polarization Field

\[ D = \varepsilon_0 E + P \]

\[ P = \text{electric flux density induced by } E \]

\[ P = \varepsilon_0 \chi_e E, \quad (4.84) \]

where \( \chi_e \) is called the *electric susceptibility* of the material. Inserting Eq. (4.84) into Eq. (4.83), we have

\[ D = \varepsilon_0 E + \varepsilon_0 \chi_e E \]

\[ = \varepsilon_0 (1 + \chi_e) E = \varepsilon E, \quad (4.85) \]
The dielectric strength $E_{ds}$ is the largest magnitude of $E$ that the material can sustain without breakdown.

**Table 4-2:** Relative permittivity (dielectric constant) and dielectric strength of common materials.

<table>
<thead>
<tr>
<th>Material</th>
<th>Relative Permittivity, $\varepsilon_r$</th>
<th>Dielectric Strength, $E_{ds}$ (MV/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air (at sea level)</td>
<td>1.0006</td>
<td>3</td>
</tr>
<tr>
<td>Petroleum oil</td>
<td>2.1</td>
<td>12</td>
</tr>
<tr>
<td>Polystyrene</td>
<td>2.6</td>
<td>20</td>
</tr>
<tr>
<td>Glass</td>
<td>4.5–10</td>
<td>25–40</td>
</tr>
<tr>
<td>Quartz</td>
<td>3.8–5</td>
<td>30</td>
</tr>
<tr>
<td>Bakelite</td>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>Mica</td>
<td>5.4–6</td>
<td>200</td>
</tr>
</tbody>
</table>

$\varepsilon = \varepsilon_r \varepsilon_0$ and $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m.
Table 4-1: Conductivity of some common materials at 20°C.

<table>
<thead>
<tr>
<th>Material</th>
<th>Conductivity, $\sigma$ (S/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Conductors</strong></td>
<td></td>
</tr>
<tr>
<td>Silver</td>
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<tr>
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</tr>
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<tr>
<td>Carbon</td>
<td>$3 \times 10^4$</td>
</tr>
<tr>
<td><strong>Semiconductors</strong></td>
<td></td>
</tr>
<tr>
<td>Pure germanium</td>
<td>2.2</td>
</tr>
<tr>
<td>Pure silicon</td>
<td>$4.4 \times 10^{-4}$</td>
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<tr>
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<tr>
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<td>Fused quartz</td>
<td>$10^{-17}$</td>
</tr>
</tbody>
</table>

Note how wide the range is, over 24 orders of magnitude.
Dielectric Breakdown

There is a limit to insulation ability of a dielectric under action of an applied electric field.

\[ E_{\text{ds}} = \text{largest electric field before "breakdown"} \]

\[ 1\text{e, dielectric} \rightarrow \text{conductor} \]

\begin{tabular}{|c|c|}
\hline
material & \( E_{\text{ds}} \) (kV/cm) \\
\hline
air & 3 \rightarrow 30kV/cm \\
oil & 12 \\
glass & 25 - 40 \\
mica & 200 \\
\hline
\end{tabular}

\[ E_{\text{field}} \sim \frac{V}{d} \]

For \( E_{\text{field}} > E_{\text{ds}} \) - lightning

breakdown biggest issue in integrated circuits!
Suppose insulator (say glass for now) \( E_0 = 10 \) V/m \( \Rightarrow \Phi = 25 \) MV/m

- Dielectric breakdown is \( 25 \times 10^6 \) V/m
  - At 1 \( \mu \)m: \( V = 25 \times 10^6 \frac{V}{m} \times 10^{-6} \text{ m} = 25 \) volts
  - At 0.1 \( \mu \)m: \( 100 \text{ nm} = 2.5 \) volts
  - At 10 nm: \( 0.25 \) volts

People constantly developing \( \Phi \) (or \( \Phi \) value) in materials, for example, if one needs more rather than glass.

\( E_0 \) is 10x that of glass so voltage to breakdown is hyper.

**BUT** can we make in device fab? Unlikely!

Also geometry effects breakdown voltage:

- Tall parallel plate: \( V = \frac{E \cdot d}{d_{	ext{min}}/2d} \)

What if a sharp tip on the cathode?

\( \Phi \ll d \)

- Can get very high fields at sharp tip (which can be atoms)

And rough at of field at sharp tip, so much higher.

His pt change Ed = \( \frac{1}{r} \), for red then even though not enough to prevent dielectric breakdown.

\( E = \Phi d \)

\( E \ll d \Rightarrow \Phi \), so boundary between insulators limited important.

In device operation, \( \Phi \) and \( d \) other.
Boundary Conditions

**Figure 4-18:** Interface between two dielectric media.

\[ E_{1t} = E_{2t} \quad (\text{V/m}) \quad \text{(4.90)} \]

\[ \hat{n}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \quad (\text{C/m}^2) \]

\[ \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2} \quad \text{(4.91)} \]

\[ D_{1n} - D_{2n} = \rho_s \quad (\text{C/m}^2) \quad \text{(4.94)} \]

The normal component of \( \mathbf{D} \) changes abruptly at a charged boundary between two different media in an amount equal to the surface charge density.
Electrostatic Boundary Conditions

\[ \vec{D} = \varepsilon \vec{E} \]

**tangential**

\[ E_{1n} \quad E_{1t} \quad E_{2n} \quad E_{2t} \]

\[ E_{1} \quad E_{2} \]

\[ d \quad \Delta h \quad \Delta h \]

\[ a \quad b \quad c \quad d \]

\[ 0 \quad 1 \]

\[ E_{1} \quad E_{2} \]

\[ E_{1n} \quad E_{1t} \quad E_{2n} \quad E_{2t} \]

\[ E_{1} \quad E_{2} \]

\[ d \quad \Delta h \quad \Delta h \]

\[ a \quad b \quad c \quad d \]

\[ 0 \quad 1 \]

\[ E_{1} \quad E_{2} \]

\[ E_{1n} \quad E_{1t} \quad E_{2n} \quad E_{2t} \]

\[ E_{1} \quad E_{2} \]

\[ d \quad \Delta h \quad \Delta h \]

\[ a \quad b \quad c \quad d \]

\[ 0 \quad 1 \]

\[ E_{1} \quad E_{2} \]

\[ E_{1n} \quad E_{1t} \quad E_{2n} \quad E_{2t} \]

\[ E_{1} \quad E_{2} \]

\[ d \quad \Delta h \quad \Delta h \]

\[ a \quad b \quad c \quad d \]

\[ 0 \quad 1 \]

\[ E_{1} \quad E_{2} \]

\[ E_{1n} \quad E_{1t} \quad E_{2n} \quad E_{2t} \]

\[ E_{1} \quad E_{2} \]

\[ d \quad \Delta h \quad \Delta h \]

\[ a \quad b \quad c \quad d \]

\[ 0 \quad 1 \]

\[ E_{1} \quad E_{2} \]

\[ E_{1n} \quad E_{1t} \quad E_{2n} \quad E_{2t} \]

\[ E_{1} \quad E_{2} \]

\[ d \quad \Delta h \quad \Delta h \]

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\[ d \quad \Delta h \quad \Delta h \]

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\[ 0 \quad 1 \]

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\[ E_{1n} \quad E_{1t} \quad E_{2n} \quad E_{2t} \]

\[ E_{1} \quad E_{2} \]

\[ d \quad \Delta h \quad \Delta h \]

\[ a \quad b \quad c \quad d \]
Electrostatic Boundary Conditions

since \( \mathbf{D} = \varepsilon \mathbf{E} \),\( \Rightarrow \) \( \mathbf{D}_{\text{t}} = \varepsilon \mathbf{E}_{\text{t}} \).

\[ \mathbf{D}_{\text{t}} = \frac{\mathbf{D}_{\text{z}}}{\varepsilon_1} \frac{\mathbf{D}_{\text{z}}}{\varepsilon_2} \]

- Tangential wumps
  - If \( \mathbf{D} \) are not continuous but depend on \( \varepsilon \).

Now look at normal wumps.

Use \( \int \mathbf{D} \cdot d\mathbf{s} = Q \), enclosed charge.

\[ Q = \int \mathbf{P}_s d\mathbf{s} \text{ surf. charge density} \]

\[ \mathbf{P}_s \approx \int_{\text{top}} D_1 \mathbf{n}_1 d\mathbf{s} + \int_{\text{bot}} D_2 \mathbf{n}_2 d\mathbf{s} + \int_{\text{cyl}} \]

\[ \mathbf{N}_0 \text{\ s.e.} \mathbf{n}_1 = -\mathbf{n}_2 \]

\[ \mathbf{P}_s = \mathbf{R}_1 (\mathbf{D}_1 - \mathbf{D}_2) \]

\[ \mathbf{P}_s = D_{\text{in}} - D_{\text{2N}} \text{ wumps of wumps change at boundary} \]

If no change at interface, \( \mathbf{P}_s = 0 \Rightarrow D_{\text{in}} = D_{\text{2N}} \)

\[ \varepsilon_1 E_{\text{in}} = \varepsilon_2 E_{\text{2N}} \]
Electrostatic Boundary Conditions

Inside perfect conductor
\[ \vec{E} = \vec{D} = 0 \]
\[ \therefore \vec{E}_2 = \vec{D}_2 = 0 \]

\[ \therefore E_{2n} = E_{2T} = 0, \quad D_{2n} = D_{2T} = 0 \text{ for conductor} \]

But \( E_T \) is continuous across boundary
\[ \therefore E_{1T} = E_{2T} = 0 \]

And \( \vec{P}_s = \vec{D}_{1n} - \vec{D}_{2n} \) the free charge density at boundary
\[ \therefore \vec{P}_s = \left[ \vec{D}_{1n} = \varepsilon_1 \vec{E}_{1n} \right] \]

\[ \therefore \vec{D}_1 = \varepsilon_1 \vec{E}_1 = \hat{n} \vec{P}_s, \quad \text{fields + surface of conductor!} \]

\[ \vec{E}_{\text{conducting}} \quad \text{or} \quad \vec{E}_{\text{conducting}} \]
what does this mean? 1e field & surf of cond. but 0 inside.

consider conductors between 2 dielectrics in uniform E field.

\[ E_1 E_0 = \rho_s \]
\[ \vec{E}_1 = \vec{E}_0 \]
\[ \vec{E}_1 = -\vec{E}_0 \]
\[ \text{just opp of ext field} \]

On bot surf
\[ \vec{E}_1 E_0 = -\rho_s \text{ some normal to the surf is downward} \]

- charge incl. on top
- charge incl. on bottom

these charges cause induced electric field!

so total field inside \( E = \vec{E}_3 + \vec{E}_1 = 0 \)

\[ \vec{E}_3 = -\vec{E}_0 \]

now look at a metal cm sphere placed in an ext field \( \vec{E}_0 \)

we see \( \vec{E}_1 + \text{induced everywhere.} \)

+ induced charge on top part of sphere
- in lower part to keep \( \vec{E} = 0 \) inside.

This is what field lines will look like.

\[ \vec{E} = -\nabla V / E \text{ for equiv.} \]

Also, what equipotential lines look like? \[ \vec{E} = -\nabla V / E \text{ for equiv.} \]

Remember
Boundary Conditions

Figure 4-18: Interface between two dielectric media.

\[ E_{1t} = E_{2t} \quad (V/m). \quad (4.90) \]

\[ \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}. \quad (4.91) \]

\[ \hat{n}_2 \cdot (D_1 - D_2) = \rho_s \quad (C/m^2). \]

\[ D_{1n} - D_{2n} = \rho_s \quad (C/m^2). \quad (4.94) \]

The normal component of \( D \) changes abruptly at a charged boundary between two different media in an amount equal to the surface charge density.
Summary of Boundary Conditions

Table 4-3: Boundary conditions for the electric fields.

<table>
<thead>
<tr>
<th>Field Component</th>
<th>Any Two Media</th>
<th>Medium 1 Dielectric $\varepsilon_1$</th>
<th>Medium 2 Conductor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tangential $E$</td>
<td>$E_{1t} = E_{2t}$</td>
<td>$E_{1t} = E_{2t} = 0$</td>
<td></td>
</tr>
<tr>
<td>Tangential $D$</td>
<td>$D_{1t}/\varepsilon_1 = D_{2t}/\varepsilon_2$</td>
<td>$D_{1t} = D_{2t} = 0$</td>
<td></td>
</tr>
<tr>
<td>Normal $E$</td>
<td>$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$</td>
<td>$E_{1n} = \rho_s/\varepsilon_1$</td>
<td>$E_{2n} = 0$</td>
</tr>
<tr>
<td>Normal $D$</td>
<td>$D_{1n} - D_{2n} = \rho_s$</td>
<td>$D_{1n} = \rho_s$</td>
<td>$D_{2n} = 0$</td>
</tr>
</tbody>
</table>

Notes: (1) $\rho_s$ is the surface charge density at the boundary; (2) normal components of $E_1$, $D_1$, $E_2$, and $D_2$ are along $\hat{n}_2$, the outward normal unit vector of medium 2.

Remember $E = 0$ in a good conductor
CD Module 4.2 Charges in Adjacent Dielectrics In two adjoining half-planes with selectable permittivities, the user can place point charges anywhere in space and select their magnitudes and polarities. The module then displays $E$, $V$, and the equipotential contours of $V$. 
CD Module 4.2 Charges in Adjacent Dielectrics In two adjoining half-planes with selectable permittivities, the user can place point charges anywhere in space and select their magnitudes and polarities. The module then displays $E$, $V$, and the equipotential contours of $V$. 
Field Lines at Conductor Boundary

At conductor boundary, $E$ field direction is always perpendicular to conductor surface.

**Figure 4-21:** Metal sphere placed in an external electric field $E_0$. 

At conductor boundary, $E$ field direction is always perpendicular to conductor surface.
CD Module 4.3 Charges above a Conducting Plane

When electric charges are placed in a dielectric medium adjoining a conducting plane, some of the conductor’s electric charges move to its surface boundary, thereby satisfying the boundary conditions outlined in Table 4-3. This module displays \( \mathbf{E} \) and \( V \) everywhere and \( \rho_s \) along the dielectric-conductor boundary.
**CD Module 4.4 Charges near a Conducting Sphere**

This module is similar to Module 4.3, except that now the conducting body is a sphere of selectable size.
remember we showed last time that no Q or E within a conductor—all charge on outside.

Let's look at a setup where V across two conductors separated by "stuff."

\[ V = \text{constant} \]

\[ \vec{E} \text{ field between the 2 charge distributions.} \]

Change on the conductors is:

\[ Q = \int_{S_1} \sigma \, dS \quad \text{surf charge density} \]

But Gauss tells us that \( Q = \int_{S} \vec{D} \cdot d\vec{S} \) and charge \( \vec{D} = \varepsilon \vec{E} \)

\[ \therefore Q = \int_{S} \varepsilon \vec{E} \cdot d\vec{S} \]

we also found that \( V = \int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{r} \) between \#1, \#2 along any line path

\[ \therefore \frac{C}{V} = \frac{\int_{S} \varepsilon \vec{E} \cdot d\vec{S}}{-\int_{S} \varepsilon \vec{E} \cdot d\vec{S}} \]

to be consistent, surf d\vec{S} in that of the + charged conductors

* NOTE: since \( \vec{E} \) is only dependent on the geometry and shape, not between conductors
Remember \( C = \frac{Q}{V} \)

\[ E_n = \hat{E} - E = \frac{p_s}{e} \text{ for conducting surf.} \]

\[ Q = \int_S p_s \, ds = \int_S \varepsilon \hat{E} \cdot E \, ds = \int_S \varepsilon \hat{E} \cdot d\vec{E} \]

Total charge on surf.

Voltage: \[ V = V_{12} = -\int_{\partial S_2} \hat{p}_1 \cdot d\vec{E} \]

\[ \therefore \quad C = \frac{Q}{V} = \frac{\int_S \varepsilon \hat{E} \cdot d\vec{E}}{-\int_{\partial S_2} \hat{p}_1 \cdot d\vec{E}} \]

Material between do.

What if material between conductors is not perfect insulator (ie, can conduct some current)? Then there will be a resistance (time).

\[ R = \frac{V}{I} = \frac{-\int_{\partial S_2} \hat{p}_1 \cdot d\vec{E}}{\int_S \varepsilon \hat{E} \cdot d\vec{E}} = \frac{-\int_{\partial S_2} \hat{p}_1 \cdot d\vec{E}}{\int_S \varepsilon \hat{E} \cdot d\vec{E}} = R \]

\[ \vec{J} = \sigma \vec{E} \]

Current density through surf. \( S \)

\[ \int_S \varepsilon \hat{E} \cdot d\vec{E} = \sigma = \frac{1}{R \sigma} \rightarrow \quad C = \frac{\sigma}{R \sigma} \]

\[ RC = \frac{\sigma}{\sigma} \]

Basis for getting capacitors from resistive conductive paper

\[ C = \frac{1}{R} \]
Capacitance

When a conductor has excess charge, it distributes the charge on its surface in such a manner as to maintain a zero electric field everywhere within the conductor, thereby ensuring that the electric potential is the same at every point in the conductor.

The capacitance of a two-conductor configuration is defined as

\[ C = \frac{Q}{V} \quad \text{(C/V or F),} \quad (4.105) \]

**Figure 4-23:** A dc voltage source connected to a capacitor composed of two conducting bodies.
Capacitance

For any two-conductor configuration:

\[ C = \frac{\int \varepsilon \mathbf{E} \cdot d\mathbf{s}}{\int_{l} \mathbf{E} \cdot d\mathbf{l}} \]  \( (\text{F}) \),

For any resistor:

\[ R = \frac{-\int_{l} \mathbf{E} \cdot d\mathbf{l}}{\int_{s} \sigma \mathbf{E} \cdot d\mathbf{s}} \]  \( (\Omega) \).  \( (4.110) \)

For a medium with uniform \( \sigma \) and \( \varepsilon \), the product of Eqs. (4.109) and (4.110) gives

\[ RC = \frac{\varepsilon}{\sigma} \]  \( (4.111) \)

This simple relation allows us to find \( R \) if \( C \) is known, or vice versa.
Example 4-11: Capacitance and Breakdown Voltage of Parallel-Plate Capacitor

\[ V = -\int_{0}^{d} \mathbf{E} \cdot d\mathbf{l} = -\int_{0}^{d} (-\hat{z}E) \cdot \hat{z} \, dz = Ed, \quad (4.112) \]

and the capacitance is

\[ C = \frac{Q}{V} = \frac{Q}{Ed} = \frac{\varepsilon A}{d}, \quad (4.113) \]

where use was made of the relation \( E = Q/\varepsilon A \).

From \( V = Ed \), as given by Eq. (4.112), \( V = V_{br} \) when \( E = E_{ds} \), the dielectric strength of the material. According to Table 4-2, \( E_{ds} = 30 \text{ (MV/m)} \) for quartz. Hence, the breakdown voltage is

\[ V_{br} = E_{ds}d = 30 \times 10^6 \times 10^{-2} = 3 \times 10^5 \text{ V}. \]
Application of Gauss's law gives:

\[ E = -\hat{r} \frac{Q}{2\pi \varepsilon rl}. \]

The potential difference \( V \) between the outer and inner conductors is

\[
V = - \int_{a}^{b} \mathbf{E} \cdot d\mathbf{l} = - \int_{a}^{b} \left( -\hat{r} \frac{Q}{2\pi \varepsilon rl} \right) \cdot (\hat{r} \, dr)
= \frac{Q}{2\pi \varepsilon l} \ln \left( \frac{b}{a} \right). \tag{4.115}
\]

The capacitance \( C \) is then given by

\[
C = \frac{Q}{V} = \frac{2\pi \varepsilon l}{\ln(b/a)}, \tag{4.116}
\]

and the capacitance per unit length of the coaxial line is

\[
C' = \frac{C}{l} = \frac{2\pi \varepsilon}{\ln(b/a)} \quad \text{(F/m)}.
\tag{4.117}
\]

\( Q \) is total charge on inside of outer cylinder, and \(-Q\) is on outside surface of inner cylinder.
For a traditional parallel-plate capacitor, what is the maximum attainable energy density?

Mica has one of the highest dielectric strengths \( \sim 2 \times 10^{8} \) V/m. If we select a voltage rating of 1 V and a breakdown voltage of 2 V (50% safety), this will require that \( d \) be no smaller than 10 nm. For mica, \( \varepsilon = 6\varepsilon_0 \) and \( \rho = 3 \times 10^{3} \) kg/m\(^3\).

Hence:

\[
W' = 90 \text{ J/kg} = 2.5 \times 10^{-2} \text{ Wh/kg}.
\]

By comparison, a lithium-ion battery has \( W' = 1.5 \times 10^{2} \) Wh/kg, almost 4 orders of magnitude greater.

Energy density is given by:

\[
W' = \frac{\varepsilon V^2}{2\rho d^2} \quad \text{(J/kg)}
\]

\( \varepsilon \) = permittivity of insulation material
\( V \) = applied voltage
\( \rho \) = density of insulation material
\( d \) = separation between plates
Electrostatic Potential Energy

Electrostatic potential energy density (Joules/volume)

\[ w_e = \frac{W_e}{V} = \frac{1}{2} \varepsilon E^2 \quad (\text{J/m}^3). \]

Energy stored in a capacitor

\[ W_e = \frac{1}{2} CV^2 \quad (\text{J}). \]

Total electrostatic energy stored in a volume

\[ W_e = \frac{1}{2} \int_V \varepsilon E^2 \, dV \quad (\text{J}). \]
Energy Stored in a Capacitor

\[ V = \frac{Q}{C} \]

\( \text{work to move an extra } dq \)

\[ d\text{We} = Vdq = \frac{Q}{C} dq \]

\[ \therefore \text{We} = \int_0^Q \frac{Q}{C} dq = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} CV^2 = \text{We} \text{ in joules} \]

\[ \text{energy stored/mass} = \frac{\text{We}}{m} = \frac{1}{2m} CV^2 \]

\( \text{if we ignore mass of condenser and assume} \)

\( m = \text{dielecrotic mass then for parallel plate capacitor} \)

\[ m = \rho \cdot \text{volume of dielectric} \]

\[ \therefore \text{We} = \frac{1}{2} \frac{Q^2}{C} \text{V}^2, \quad C = \frac{\epsilon A}{d} \rightarrow \text{II plate capacit} \]

\[ \frac{\text{We}}{m} = \frac{\epsilon V^2}{2 \rho d^2} \text{ joules/kg} \]
Tech Brief 8: **Supercapacitors**

For a traditional parallel-plate capacitor, what is the maximum attainable energy density?

Energy density is given by:

\[
W' = \frac{\varepsilon V^2}{2\rho d^2} \quad (\text{J/kg})
\]

- \(\varepsilon\) = permittivity of insulation material
- \(V\) = applied voltage
- \(\rho\) = density of insulation material
- \(d\) = separation between plates

Mica has one of the highest dielectric strengths \(\sim 2 \times 10^{8} \text{ V/m}\).

If we select a voltage rating of 1 V and a breakdown voltage of 2 V (50% safety), this will require that \(d\) be no smaller than 10 nm.

For mica, \(\varepsilon = 6\varepsilon_0\) and \(\rho = 3 \times 10^{3} \text{ kg/m}^3\).

Hence:

\[
W' = 90 \text{ J/kg} = 2.5 \times 10^{-2} \text{ Wh/kg}.
\]

By comparison, a lithium-ion battery has \(W' = 1.5 \times 10^{2} \text{ Wh/kg},\) almost 4 orders of magnitude greater.
A supercapacitor is a “hybrid” battery/capacitor

Figure TF8-1: Cross-sectional view of an electrochemical double-layer capacitor (EDLC), otherwise known as a supercapacitor. (Courtesy of Ultracapacitor.org.)
Users of Supercapacitors

Figure TF8-2: Examples of systems that use supercapacitors. (Courtesy of Railway Gazette International; BMW; NASA; Applied Innovative Technologies.)
Energy Comparison

Energy Storage Devices

<table>
<thead>
<tr>
<th>Feature</th>
<th>Traditional Capacitor</th>
<th>Supercapacitor</th>
<th>Battery</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy density $W'$ (Wh/kg)</td>
<td>$\sim 10^{-2}$</td>
<td>1 to 10</td>
<td>5 to 150</td>
</tr>
<tr>
<td>Power density $P'$ (W/kg)</td>
<td>1,000 to 10,000</td>
<td>1,000 to 5,000</td>
<td>10 to 500</td>
</tr>
<tr>
<td>Charge and discharge rate $T$</td>
<td>$10^{-3}$ sec</td>
<td>$\sim$ 1 sec to 1 min</td>
<td>$\sim$ 1 to 5 hrs</td>
</tr>
<tr>
<td>Cycle life $N_c$</td>
<td>$\infty$</td>
<td>$\sim 10^6$</td>
<td>$\sim 10^3$</td>
</tr>
</tbody>
</table>

Figure TF8-3: Comparison of energy storage devices.
Electrostatic Potential Energy

Electrostatic potential energy density (Joules/volume)

$$w_e = \frac{W_e}{V} = \frac{1}{2} \varepsilon E^2 \quad (\text{J/m}^3).$$

Energy stored in a capacitor

$$W_e = \frac{1}{2} CV^2 \quad (\text{J}).$$

Total electrostatic energy stored in a volume

$$W_e = \frac{1}{2} \int_V \varepsilon E^2 \, dV \quad (\text{J})$$
Image Method

![Diagram showing an electric field configuration with a grounded perfectly conducting plane.](image)

(a) Charge $Q$ above grounded plane

(b) Equivalent configuration

**Figure 4-26:** By image theory, a charge $Q$ above a grounded perfectly conducting plane is equivalent to $Q$ and its image $-Q$ with the ground plane removed.

Image method simplifies calculation for $E$ and $V$ due to charges near conducting planes.

1. For each charge $Q$, add an image charge $-Q$
2. Remove conducting plane
3. Calculate field due to all charges
Image Method

Figure 4-26: By image theory, a charge $Q$ above a grounded perfectly conducting plane is equivalent to $Q$ and its image $-Q$ with the ground plane removed.

**Image method simplifies calculation for $E$ and $V$ due to charges near conducting planes.**

1. For each charge $Q$, add an image charge $-Q$
2. Remove conducting plane
3. Calculate field due to all charges
**Fluid Gauge**

The two metal electrodes in Fig. TF9-1(a), usually rods or plates, form a capacitor whose capacitance is directly proportional to the *permittivity* of the material between them. If the fluid section is of height $h_f$ and the height of the empty space above it is $(h - h_f)$, then the overall capacitance is equivalent to two capacitors in parallel, or

$$C = C_f + C_a = \varepsilon_f w \frac{h_f}{d} + \varepsilon_a w \frac{(h - h_f)}{d},$$

where $w$ is the electrode plate width, $d$ is the spacing between electrodes, and $\varepsilon_f$ and $\varepsilon_a$ are the permittivities of the fluid and air, respectively. Rearranging the expression as a linear equation yields

$$C = kh_f + C_0,$$
Humidity Sensor

Figure TF9-2: Interdigital capacitor used as a humidity sensor.
Pressure Sensor

Figure TF9-3: Pressure sensor responds to deflection of metallic membrane.
Pressure Sensor

Figure TF9-3: Pressure sensor responds to deflection of metallic membrane.
Planar capacitors

**Figure TF9-4:** Concentric-plate capacitor.

**Figure TF9-5:** (a) Adjacent-plates capacitor; (b) perturbation field.
Fingerprint Imager

Figure TF9-6: Elements of a fingerprint matching system. (Courtesy of IEEE Spectrum.)

Figure TF9-7: Fingerprint representation. (Courtesy of Dr. M. Tartagni, University of Bologna, Italy.)
**Chapter 4 Relationships**

Maxwell’s Equations for Electrostatics

<table>
<thead>
<tr>
<th>Name</th>
<th>Differential Form</th>
<th>Integral Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauss’s law</td>
<td>( \nabla \cdot \mathbf{D} = \rho_s )</td>
<td>( \oint \mathbf{D} \cdot d\mathbf{s} = Q )</td>
</tr>
<tr>
<td>Kirchhoff’s law</td>
<td>( \nabla \times \mathbf{E} = 0 )</td>
<td>( \oint \mathbf{E} \cdot d\mathbf{l} = 0 )</td>
</tr>
</tbody>
</table>

**Electric Field**

| Current density \( \mathbf{J} = \rho_s \mathbf{u} \) | Point charge | \( \mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi \varepsilon R^2} \) |
| Poisson’s equation | \( \nabla^2 V = -\frac{\rho_s}{\varepsilon} \) | Many point charges | \( \mathbf{E} = \frac{1}{4\pi \varepsilon} \sum_{i=1}^{N} \frac{q_i (\mathbf{R} - \mathbf{R}_i)}{|\mathbf{R} - \mathbf{R}_i|^3} \) |
| Laplace’s equation | \( \nabla^2 V = 0 \) | Volume distribution | \( \mathbf{E} = \frac{1}{4\pi \varepsilon} \int_{V'} \mathbf{\hat{R}}' \frac{\rho_s dV'}{R'^2} \) |
| Resistance | \( R = \frac{\int \mathbf{E} \cdot d\mathbf{l}}{\oint_s \sigma \mathbf{E} \cdot d\mathbf{s}} \) | Surface distribution | \( \mathbf{E} = \frac{1}{4\pi \varepsilon} \int_{S'} \mathbf{\hat{R}}' \frac{\rho_s ds'}{R'^2} \) |
| Boundary conditions | Table 4-3 | Line distribution | \( \mathbf{E} = \frac{1}{4\pi \varepsilon} \int_{l'} \mathbf{\hat{R}}' \frac{\rho_e dl'}{R'^2} \) |
| Capacitance | \( C = \frac{\oint_s \sigma \mathbf{E} \cdot d\mathbf{s}}{-\int_l \mathbf{E} \cdot d\mathbf{l}} \) | Infinite sheet of charge | \( \mathbf{E} = \hat{\mathbf{z}} \frac{\rho_s}{2\varepsilon_0} \) |
| \( RC \) relation | \( RC = \frac{\varepsilon}{\sigma} \) | Infinite line of charge | \( \mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0} = \hat{\mathbf{r}} \frac{D_r}{\varepsilon_0} = \hat{\mathbf{r}} \frac{\rho_e}{2\pi \varepsilon_0 r} \) |
| Energy density \( w_e = \frac{1}{2} \varepsilon E^2 \) | Dipole | \( \mathbf{E} = \frac{qd}{4\pi \varepsilon_0 R^3} (\hat{\mathbf{R}} \cos \theta + \hat{\mathbf{\theta}} \sin \theta) \) |
| Relation to \( V \) | \( \mathbf{E} = -\nabla V \) | | |