The U.S.-Denmark Summer Workshop takes place annually in California and Denmark alternately, with the 2013 edition in Denmark. The four-week workshop starts with one week of online preparation and continues with three weeks of lectures, seminars and field trips to renewable energy sites and facilities in Denmark: providing students with real-world experience of the technological and social aspects of RE implementation at a local level. The faculty is composed of U.S. and Danish professors, as well as, external professionals and researchers with proven experience in their field. Students will work on team-based projects related to renewable energy solutions to specific problems.

Applications Due 3/8/2013
For details on how to apply go to:
https://pire.soe.ucsc.edu/2013/summer

This course is worth 7 credit units and is offered through UC Santa Cruz Summer Session. Financial aid fellowships are available for current qualified students who are U.S. citizens or permanent residents.

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Funding provided by NSF–PIRE Award #1243536
EE 135, Winter 2013

Reading: Chapter 4. chapter 4.10-4.11, chapter 5.1-5.4

Homework #4, due 2/14: Chapter 4, problems, 4.10, 4.21, 4.24, 4.31, 4.46

For laboratory: read lab. introduction before lab.

NOTE: Midterm# 2 is on **February 28**, not February 21.

Lecture 10
Electrostatic Potential Energy

Electrostatic potential energy density (Joules/volume)

\[ w_e = \frac{W_e}{V} = \frac{1}{2} \varepsilon E^2 \quad (\text{J/m}^3). \]

Energy stored in a capacitor

\[ W_e = \frac{1}{2} CV^2 \quad (\text{J}). \]

Total electrostatic energy stored in a volume

\[ W_e = \frac{1}{2} \int_V \varepsilon E^2 \, dV \quad (\text{J}) \]
Energy Stored in a Capacitor

\[ V = \frac{Q}{C} \]

work to move an extra dq

\[ dW_e = Vdq = \frac{Q}{C}dq \]

\[ : W_e = \int_0^Q \frac{Q}{C}dq = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} CV^2 = W_e \text{ joules} \]

energy stored/mass = \[ \frac{W_e}{m} = \frac{1}{2m} CV^2 \]

If we ignore mass of conductors and assume
\[ m = \text{deleterious mass then for parallel plate capac.} \]
\[ m = pAd \rightarrow \text{ density of positive } \]
\[ \therefore \frac{W_e}{m} = \frac{1}{2pAd} (c) V^2, \quad c = \frac{EA}{d} \rightarrow \text{ II plate capac.} \]

\[ \frac{W_e}{m} = \frac{EV^2}{2pd^2} \text{ joules/kg} \]
# Energy Comparison

## Energy Storage Devices

<table>
<thead>
<tr>
<th>Feature</th>
<th>Traditional Capacitor</th>
<th>Supercapacitor</th>
<th>Battery</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy density $W'$ (Wh/kg)</td>
<td>$\sim 10^{-2}$</td>
<td>1 to 10</td>
<td>5 to 150</td>
</tr>
<tr>
<td>Power density $P'$ (W/kg)</td>
<td>1,000 to 10,000</td>
<td>1,000 to 5,000</td>
<td>10 to 500</td>
</tr>
<tr>
<td>Charge and discharge rate $T$</td>
<td>$10^{-3}$ sec</td>
<td>~ 1 sec to 1 min</td>
<td>~ 1 to 5 hrs</td>
</tr>
<tr>
<td>Cycle life $N_c$</td>
<td>$\infty$</td>
<td>$\sim 10^6$</td>
<td>$\sim 10^3$</td>
</tr>
</tbody>
</table>

Figure TF8-3: Comparison of energy storage devices.
Image Method

Figure 4-26: By image theory, a charge $Q$ above a grounded perfectly conducting plane is equivalent to $Q$ and its image $-Q$ with the ground plane removed.

Image method simplifies calculation for $E$ and $V$ due to charges near conducting planes.

1. For each charge $Q$, add an image charge $-Q$
2. Remove conducting plane
3. Calculate field due to all charges
Example 4-13: Image Method for Charge Above Conducting Plane

Use image theory to determine $\mathbf{E}$ at an arbitrary point $P = (x, y, z)$ in the region $z > 0$ due to a charge $Q$ in free space at a distance $d$ above a grounded conducting plate residing in the $z = 0$ plane.

**Solution:** In Fig. 4-28, charge $Q$ is at $(0, 0, d)$ and its image $-Q$ is at $(0, 0, -d)$. From Eq. (4.19), the electric field at point $P = (x, y, z)$ due to the two charges is given by

$$\mathbf{E} = \frac{1}{4\pi \varepsilon_0} \left( \frac{QR_1}{R_1^3} + \frac{-QR_2}{R_2^3} \right)$$

$$= \frac{Q}{4\pi\varepsilon_0} \left[ \frac{\hat{x}x + \hat{y}y + \hat{z}(z - d)}{[x^2 + y^2 + (z - d)^2]^{3/2}} \right] - \frac{Q}{4\pi\varepsilon_0} \left[ \frac{\hat{x}x + \hat{y}y + \hat{z}(z + d)}{[x^2 + y^2 + (z + d)^2]^{3/2}} \right]$$

for $z \geq 0$.

**Figure 4-28:** Application of the image method for finding $\mathbf{E}$ at point $P$ (Example 4-13).
Problem 4.63

\[ \text{ground plane} \quad \nabla v = 0 \]

Assume uniform charge density on cylinder ($\pm \text{page, } \infty \text{ lmg}$), $\rho_e = (\text{cnl/m})$

Use "image charge" method to get at $E$ field and potential.

Calculate potential at $z$ above ground plane along line between centers of cylinders!!

Why will this configuration give no proper $E$ field?
Example 4-7: Electric Field of an Electric Dipole (cont.)

\[ qd \cos \theta = qd \cdot \hat{R} = p \cdot \hat{R}, \]

where \( p = qd \) is called the dipole moment. Using Eq. (4.53) in Eq. (4.52) then gives

\[ V = \frac{p \cdot \hat{R}}{4\pi \varepsilon_0 R^2} \quad \text{(electric dipole).} \quad (4.54) \]

In spherical coordinates, Eq. (4.51) is given by

\[
E = -\nabla V \\
= - \left( \hat{R} \frac{\partial V}{\partial R} + \hat{\theta} \frac{1}{R} \frac{\partial V}{\partial \theta} + \hat{\phi} \frac{1}{R \sin \theta} \frac{\partial V}{\partial \phi} \right), \quad (4.55)
\]

\[ E = \frac{qd}{4\pi \varepsilon_0 R^3} \left( 2\cos \theta + \hat{\theta} \sin \theta \right) \quad \text{(V/m)}. \]
Problem 4.63

Remember, \( E \) from a line charge, density \( \rho \)

\[
E = \frac{\rho}{2\pi \varepsilon_0 r}
\]

\( r \) = radial dist. from line.

If it is a cyl conductor of radius \( a \)
then same formula holds for \( r \geq a \).

\[
E_p = \frac{r \rho}{2\pi \varepsilon_0 r}
\]
Problem 4.63 (wt)

we can calculate potential along any path between conducting cylinders and "ground" plane.

. choose simplest path - along a line connecting them.

\[ E_{up} = -\frac{2}{\pi} \frac{P_0}{2\pi d(d-z)} \]  
assumes air between them

\[ E_{low} = \frac{2}{\pi} \frac{(-P_0)}{2\pi d(d+z)} \]  
dist. from centers

at \( V=0 \) plane, dist. to centers of upper "real" cylinder.

at \( V=0 \) plane, dist. to centers of lower "virtual" cylinder.
Problem 4.63 (unit)

\[ E_{up} = -\frac{2}{\pi \varepsilon_0} \frac{\rho E}{(d^2 - z^2)} \]

real

\[ E_{down} = -\frac{2}{\pi \varepsilon_0} \frac{\rho E}{(d^2 + z^2)} \]

virtual

potential between the 2 cylinders.

\[ V = -\int_{(d-a)}^{a} E \cdot dz \]

\[ = -\int_{(d-a)}^{a} \frac{\rho E}{2 \pi \varepsilon_0} \left[ \frac{1}{d^2 - z^2} + \frac{1}{d^2 + z^2} \right] \cdot dz \]

\[ = \frac{\rho E}{2 \pi \varepsilon_0} \int_{(d-a)}^{a} \left( \frac{1}{d^2 - z^2} + \frac{1}{d^2 + z^2} \right) dz \]

\[ = \frac{\rho E}{2 \pi \varepsilon_0} \left[ -\ln(d-z) + \ln(d+z) \right]_{(d-a)}^{(d-a)} \]

\[ = \frac{\rho E}{2 \pi \varepsilon_0} \left[ -\ln(a) + \ln(2d-a) + \ln(2d-a) - \ln(a) \right] \]

\[ V = \frac{\rho E}{2 \pi \varepsilon_0} \ln \left( \frac{2d-a}{a} \right) \]

between the 2 cylinders.
Problem 4.63 (cont)

.. for a length $L$, $Q = p_e L$
we have calculate for $L \to \infty$

so $C = \frac{Q}{V} = \frac{p_e L}{\frac{P_e}{V E_0} \ln \left( \frac{2d-a}{a} \right)}$

\[ C = \frac{P_e \ln \left( \frac{2d-a}{a} \right)}{V E_0} \]

\[ \frac{C}{L} = \frac{\pi e_0}{\ln \left( \frac{2d-a}{a} \right)} \]

$d$ = dist. to ground pln.
$a$ = rad. of cylinders.

Circuit/length of conducting cyl above a ground plane.
Tech Brief 9: Capacitive Sensors

Fluid Gauge

The two metal electrodes in Fig. TF9-1(a), usually rods or plates, form a capacitor whose capacitance is directly proportional to the permittivity of the material between them. If the fluid section is of height $h_f$ and the height of the empty space above it is $(h - h_f)$, then the overall capacitance is equivalent to two capacitors in parallel, or

$$C = C_f + C_a = \varepsilon_f w \frac{h_f}{d} + \varepsilon_a w \frac{(h - h_f)}{d},$$

where $w$ is the electrode plate width, $d$ is the spacing between electrodes, and $\varepsilon_f$ and $\varepsilon_a$ are the permittivities of the fluid and air, respectively. Rearranging the expression as a linear equation yields

$$C = k h_f + C_0,$$

where $C_0 =$ capacitance when empty.
Humidity Sensor

Figure TF9-2: Interdigital capacitor used as a humidity sensor.
Pressure Sensor

Figure TF9-3: Pressure sensor responds to deflection of metallic membrane.
Planar capacitors

**Figure TF9-4:** Concentric-plate capacitor.

**Figure TF9-5:** (a) Adjacent-plates capacitor; (b) perturbation field.
Fingerprint Imager

Figure TF9-6: Elements of a fingerprint matching system. (Courtesy of IEEE Spectrum.)

Figure TF9-7: Fingerprint representation. (Courtesy of Dr. M. Tartagni, University of Bologna, Italy.)
Chapter 4 Relationships

Maxwell’s Equations for Electrostatics

<table>
<thead>
<tr>
<th>Name</th>
<th>Differential Form</th>
<th>Integral Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauss’s law</td>
<td>$\nabla \cdot D = \rho_v$</td>
<td>$\oint_S D \cdot ds = Q$</td>
</tr>
<tr>
<td>Kirchhoff’s law</td>
<td>$\nabla \times E = 0$</td>
<td>$\oint_C E \cdot dl = 0$</td>
</tr>
</tbody>
</table>

Current density $J = \rho_v \mathbf{u}$

Poisson’s equation $\nabla^2 V = -\frac{\rho_v}{\varepsilon}$

Laplace’s equation $\nabla^2 V = 0$

Resistance $R = \frac{\int_{l} E \cdot dl}{\int_{s} \varepsilon E \cdot ds}$

Boundary conditions Table 4-3

Capacitance $C = \frac{\int_{s} \varepsilon E \cdot ds}{-\int_{l} E \cdot dl}$

$RC$ relation $RC = \frac{\varepsilon}{\sigma}$

Energy density $w_e = \frac{1}{2} \varepsilon E^2$

Electric Field

Point charge $E = \hat{R} \frac{q}{4\pi \varepsilon R^2}$

Many point charges $E = \frac{1}{4\pi \varepsilon} \sum_{i=1}^{N} \frac{q_i (\mathbf{R} - \mathbf{R}_i)}{|\mathbf{R} - \mathbf{R}_i|^3}$

Volume distribution $E = \frac{1}{4\pi \varepsilon} \int_{V'} \hat{R'} \frac{\rho_v}{R^2} dV'$

Surface distribution $E = \frac{1}{4\pi \varepsilon} \int_{S'} \hat{R'} \frac{\rho_s}{R^2} ds'$

Line distribution $E = \frac{1}{4\pi \varepsilon} \int_{l'} \hat{R'} \frac{\rho_l}{R^2} dl'$

Infinite sheet of charge $E = \hat{z} \frac{\rho_s}{2\varepsilon_0}$

Infinite line of charge $E = \frac{\mathbf{D}}{\varepsilon_0} = \hat{r} \frac{D_r}{\varepsilon_0} = \hat{r} \frac{\rho_l}{2\pi \varepsilon_0 r}$

Dipole $E = \frac{qd}{4\pi \varepsilon_0 R^3} (\hat{\mathbf{R}} \cos \theta + \hat{\mathbf{0}} \sin \theta)$

Relation to $V$ $E = -\nabla V$
4 different methods for calculating Electric Fields ($\vec{E}$, $\vec{D}$)

1. Coulomb law: $\frac{d\vec{E}}{d\vec{r}} \propto \frac{d\vec{q}}{r^2}$

2. Gauss' law: $\oint \vec{E} \cdot d\vec{s} = \frac{Q_{enc}}{\varepsilon}$

3. Poisson's equation:
   $$\nabla^2 V = -\frac{\rho_V}{\varepsilon}$$

4. Image charges
   (useful for charge distributions near conducting surfaces)
Figure TF11-1: Linear variable differential transformer (LVDT) circuit.

5. MAGNETOSTATICS
# Chapter 5 Overview

## Chapter Contents

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## Objectives

Upon learning the material presented in this chapter, you should be able to:

1. Calculate the magnetic force on a current-carrying wire placed in a magnetic field and the torque exerted on a current loop.
2. Apply the Biot–Savart law to calculate the magnetic field due to current distributions.
3. Apply Ampère’s law to configurations with appropriate symmetry.
4. Explain magnetic hysteresis in ferromagnetic materials.
5. Calculate the inductance of a solenoid, a coaxial transmission line, or other configurations.
6. Relate the magnetic energy stored in a region to the magnetic field distribution in that region.
# Electric vs Magnetic Comparison

**Table 5-1:** Attributes of electrostatics and magnetostatics.

<table>
<thead>
<tr>
<th>Attribute</th>
<th>Electrostatics</th>
<th>Magnetostatics</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Sources</strong></td>
<td>Stationary charges $\rho_v$</td>
<td>Steady currents $J$</td>
</tr>
<tr>
<td><strong>Fields and Fluxes</strong></td>
<td>$E$ and $D$</td>
<td>$H$ and $B$</td>
</tr>
<tr>
<td><strong>Constitutive parameter(s)</strong></td>
<td>$\varepsilon$ and $\sigma$</td>
<td>$\mu$</td>
</tr>
<tr>
<td><strong>Governing equations</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>- Differential form</td>
<td>$\nabla \cdot D = \rho_v$</td>
<td>$\nabla \cdot B = 0$</td>
</tr>
<tr>
<td></td>
<td>$\nabla \times E = 0$</td>
<td>$\nabla \times H = J$</td>
</tr>
<tr>
<td>- Integral form</td>
<td>$\int_S D \cdot ds = Q$</td>
<td>$\int_S B \cdot ds = 0$</td>
</tr>
<tr>
<td></td>
<td>$\int_C E \cdot dl = 0$</td>
<td>$\int_C H \cdot dl = I$</td>
</tr>
<tr>
<td><strong>Potential</strong></td>
<td>Scalar $V$, with $E = -\nabla V$</td>
<td>Vector $A$, with $B = \nabla \times A$</td>
</tr>
<tr>
<td><strong>Energy density</strong></td>
<td>$w_e = \frac{1}{2} \varepsilon E^2$</td>
<td>$w_m = \frac{1}{2} \mu H^2$</td>
</tr>
<tr>
<td><strong>Force on charge $q$</strong></td>
<td>$F_e = qE$</td>
<td>$F_m = q\mathbf{u} \times \mathbf{B}$</td>
</tr>
<tr>
<td><strong>Circuit element(s)</strong></td>
<td>$C$ and $R$</td>
<td>$L$</td>
</tr>
</tbody>
</table>
Electric & Magnetic Forces

**Magnetic force**
\[ F_m = q u \times B \quad (N) \]

**Electromagnetic (Lorentz) force**
\[ F = F_e + F_m = qE + qu \times B = q(E + u \times B). \]

*Figure 5-1:* The direction of the magnetic force exerted on a charged particle moving in a magnetic field is (a) perpendicular to both \(B\) and \(u\) and (b) depends on the charge polarity (positive or negative).
Exercise 5.3

def: velocity filter

Lorentz Force:
\[ \mathbf{F} = \mathbf{F}_E + \mathbf{F}_B = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \]

assume
\[ \mathbf{E} = \hat{x} E, \quad \mathbf{B} = \hat{y} B \quad \text{and} \quad \mathbf{v} = -\hat{z} v \]

\[ \mathbf{F} = q (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \]

\[ \therefore \text{for only change of velocity } \mathbf{v} \text{ to pass through saved fields with no deflection,} \]
\[ \mathbf{E} = -\hat{z} \times \mathbf{B} \]

\[ \hat{y} \times E = (\hat{z} \times \mathbf{v}) \times \mathbf{B} = nB \hat{z} \times \hat{y} \]
\[ \hat{z} \times \hat{y} = -\hat{x} \]
\[ = -nB \hat{x} \]

\[ \therefore \mathbf{E} = 100 \times \frac{E}{B} \quad \text{m} - \text{s} \text{ duration} \]
\[ \text{velocity filter} \]
CD Module 5.1 Electron Motion in Static Fields

This module demonstrates the Lorentz force on an electron moving under the influence of an electric field alone, a magnetic field alone, or both acting simultaneously.
Charged particle Velocity Filter

Exercise 5.3

device: velocity filter

Lorentz Force:
\[ F = F_c + F_B = qE + qv \times B \]

assume
\[ E = \hat{z}E, \quad B = \hat{y}B \quad \text{and} \quad \vec{v} = -\hat{r} \vec{r} \]

\[ F \text{ on } q \parallel \vec{r} = q(E + \vec{v} \times \vec{B}) \]

\[ \therefore \text{ an only change of velocity } \vec{v} \text{ to pass through crossed fields with no deflection,} \]
\[ \vec{E} = -\vec{v} \times \vec{B} \]
\[ \therefore \vec{E} = -(-\hat{r} \vec{r}) \times \hat{y} \vec{B} = n \vec{B} \hat{z} \times \hat{y} \]
\[ = n \vec{B} \hat{z} \]
\[ \therefore |\vec{F}| = \frac{E}{B} \]
\[ \vec{E} \text{ m- } \vec{B} \text{ and } \vec{v} \text{ m- } \vec{B} \]
\[ \text{a velocity filter!} \]
No Work Done in Moving a Charged Particle in a Magnetic Field.

\[
\text{Work:} \quad dW = \vec{F}_M \cdot d\vec{r} = \vec{F}_E \cdot d\vec{l}
\]

\[
\vec{F}_E = q \vec{E} \Rightarrow dW_E = q \vec{E} \cdot d\vec{l} \quad \text{can be considered zero}
\]

but \( \vec{F}_M = q \vec{n} \times \vec{B} \)

\[
\therefore dW_M = (q \vec{n} \times \vec{B}) \cdot d\vec{l}, \quad d\vec{l} = \vec{n} dt
\]

\[
\therefore dW_M = (q \vec{n} \times \vec{B}) \cdot \vec{n} \cdot dt
\]

\[
\rightarrow \quad \boxed{dW_M = 0}
\]

The magnetic field cannot change the kinetic energy of a charged particle.
** Charged Particle in Uniform $\vec{B}$ Field**

\[ \vec{F} = q\vec{v} \times \vec{B} \]

Into page

no parallel component
since $\vec{v}$ is parallel to $\vec{B}$

\[ \therefore F_{\perp} = qvB \sin \theta \]

If $\theta = 90^\circ$, $F_{\perp} = qvB = \frac{mv^2}{R}$

particle circles about $\vec{B}$ field lines, $rad = R$

\[ \therefore \frac{mv^2}{q} = BR \]

magnetic rigidity

(like a momentum/unit charge)

units of Tesla-meters

suppose an electron and proton have same velocity —

$e^-$ will circle around $\vec{B}$ field with smaller $R$

by $\frac{m_e}{m_p} = \frac{1}{1837}$ —

3 orders of magnitude difference!!
Electron path in uniform magnetic field
Example of electron motion in $\vec{B}$ field

\[ BR = \frac{mv}{Q} \Rightarrow \sqrt{\frac{2mvT}{Q}} \rightarrow \text{kinetic energy} \]

\[ \therefore BR = \sqrt{\frac{2mc^2}{Q}} T \quad c = \text{rel. light in vacuum} \]

\[ BR = \frac{10^{-6}}{300} \sqrt{2T(m^2)} \rightarrow \text{rest energy of electron in eV} = 511 \text{keV} \]

\[ \text{Kinetic energy in eV} \]

eg. for 25 keV electrons, $T = 25 \text{keV}$

\[ BR \approx 525 \times 10^{-6} \text{ Tesla m} \]

\[ BR \approx 525 \text{ Gauss-cm} \quad 1 \text{ Gauss} = 10^{-4} \text{ Tesla} \]

Earth's field \( \approx \frac{1}{2} \text{ Gauss} \)

so 25 keV electron has \( \sim 10^3 \text{ cm rad. of curve orbit} \)
electron motion in $\vec{B}$ field (unit)

Effect of $\vec{B}$ fields on electron beam machines.

For electron $Q = -e$ so $
\vec{F} = e \vec{N} \times \vec{B}, \ \text{down!}$

from geometry

$R^2 = (R - ay)^2 + l^2$
$= R^2 - 2R ay + ay^2 + l^2$. If $ay \ll l$ then

$R^2 \approx R^2 - 2R ay + l^2$

$\frac{dy}{l^2} = \frac{By}{2BR} \rightarrow \text{deflection of e- in } \vec{B} \text{ field}$

So if $\vec{B} = 10^{-3} \text{ gauss} / (10^{-3} \text{ Tesla})$
then $ay \approx \frac{1}{2} ay^2 \text{ cm in earth's field}$
so for 10 cm length $\rightarrow ay \approx \frac{1}{2} \text{ mm}$
But AC fields, "wobble" the beam.

eq. if $B = 10^{-2} \text{ gauss} / (10^{-2} \text{ Tesla})$

$\text{ay} = 10^{-4} \text{ cm} = \text{micro cm for } l = 10 \text{ cm}$

problems for e-beam lithography and micro
Differences between Electric and Magnetic Forces

1. Electric force always in direction of electric field

2. Electric force acts on charged particle whether it is moving or not.

3. Electric force does work in acting on a charged particle

4. Magnetic force does no work since it is perpendicular to direction of movement. Therefore, a magnetic field cannot change the energy of a charged particle.
Hans Christian Ørsted: Who He Was, and Why You Owe Him

Ker Than
for National Geographic News
August 14, 2009

Hans Christian Ørsted: He's not as famous as Darwin or Newton, but if you've ever used a modern gadget, chances are you have this 19th-century Danish physicist to thank—and what better time than on his 232nd birthday?

Hans Christian Ørsted
Danish physicist & chemist

- Born: 14 August 1777, Rudkøbing, Denmark
- Died: 9 March 1851 (aged 73), Copenhagen, Denmark
- Nationality: Danish
- Fields: physics, chemistry
- Known for: electromagnetism
- Influences: Immanuel Kant

Signature:

[Signature]

[Diagram of a magnetic compass]
Magnetic Force on Current Carrying Wire

\[ \mathbf{F} = I \oint \mathbf{dA} \times \mathbf{B} \]

For a closed loop
Magnetic Force on a Current Element

Differential force $dF_m$ on a differential current $I\,dl$:

$$dF_m = I\,dl \times B \quad \text{(N).} \quad (5.9)$$

For a closed circuit of contour $C$ carrying a current $I$, the total magnetic force is

$$F_m = I \oint_C dl \times B \quad \text{(N).} \quad (5.10)$$

If the closed wire shown in Fig. 5-3(a) resides in a uniform external magnetic field $B$, then $B$ can be taken outside the integral in Eq. (5.10), in which case

$$F_m = I \left( \oint_C dl \right) \times B = 0. \quad (5.11)$$

This result, which is a consequence of the fact that the vector sum of the infinitesimal vectors $dl$ over a closed path equals zero, states that the total magnetic force on any closed current loop in a uniform magnetic field is zero.

Figure 5-2: When a slightly flexible vertical wire is placed in a magnetic field directed into the page (as denoted by the crosses), it is (a) not deflected when the current through it is zero, (b) deflected to the left when $I$ is upward, and (c) deflected to the right when $I$ is downward.
Force in a semicircular conductor. EX. 5.1

\[ \overrightarrow{dF} = I \, d\theta \times B \]

For st. section, length 2r.

\[ \overrightarrow{F_1} = \hat{r} (2\pi I) x \hat{r} \hat{B} = 2 \pi I \hat{r} \hat{B} = \overrightarrow{F_1} \text{ on st. wire} \]

Force on curved section.

\[ \overrightarrow{dF} \hat{B} \text{ with in xy plane, so } \overrightarrow{d\xi \times \hat{B}} \text{ in } -\hat{z} \text{ direction} \]

\[ |d\xi| = rd\theta \sin \phi \]

\[ \int_{\phi=0}^{\pi} r \, d\theta \, B \sin \phi \]

\[ \therefore \overrightarrow{F_2} = I \int_{\phi=0}^{\pi} d\xi \times \hat{B} = I \int_{\phi=0}^{\pi} r \, d\theta \, B \sin \phi \, (-\hat{z}) \]

\[ \overrightarrow{F_2} = -\hat{z} \pi r B \int_{0}^{\pi} d\theta \sin \phi \]

\[ \overrightarrow{F_2} = -\frac{1}{2} [2\pi I \hat{r} \hat{B}] \text{ on curved wire} \]

\[ \overrightarrow{F} = \overrightarrow{F_1} + \overrightarrow{F_2} = 0 \]

\[ \text{no net force on a closed loop} \]
Torque

\[ T = d \times F \quad \text{(N\cdot m)} \]

- \( d \) = moment arm
- \( F \) = force
- \( T \) = torque

**Figure 5-5**: The force \( F \) acting on a circular disk that can pivot along the \( z \)-axis generates a torque \( T = d \times F \) that causes the disk to rotate.

*These directions are governed by the following right-hand rule: when the thumb of the right hand points along the direction of the torque, the four fingers indicate the direction that the torque tries to rotate the body.*
Magnetic Torque on Current Loop

\[ \mathbf{F}_1 = I (\hat{y}b) \times (\hat{x}B_0) = \hat{z}IlbB_0, \]

\[ \mathbf{F}_3 = I (\hat{y}b) \times (\hat{x}B_0) = -\hat{z}IlbB_0. \]

No forces on arms 2 and 4 (because \( I \) and \( B \) are parallel, or anti-parallel)

**Magnetic torque:**

\[ \mathbf{T} = \mathbf{d}_1 \times \mathbf{F}_1 + \mathbf{d}_3 \times \mathbf{F}_3 \]

\[ = \left( -\hat{x} \frac{a}{2} \right) \times (\hat{z}IlbB_0) + \left( \hat{x} \frac{a}{2} \right) \times (-\hat{z}IlbB_0) \]

\[ = \hat{y}IabB_0 = \hat{y}IAB_0, \]

B in plane of loop

Area of Loop

**Figure 5-6:** Rectangular loop pivoted along the y-axis: (a) front view and (b) bottom view. The combination of forces \( \mathbf{F}_1 \) and \( \mathbf{F}_3 \) on the loop generates a torque that tends to rotate the loop in a clockwise direction as shown in (b).
Inclined Loop

For a loop with \( N \) turns and whose surface normal is at angle \( \theta \) relative to \( B \) direction:

\[
T = NIA B_0 \sin \theta. \quad (5.18)
\]

The quantity \( NIA \) is called the magnetic moment \( m \) of the loop. Now, consider the vector

\[
m = \hat{n} NIA = \hat{n} m \quad (\text{A} \cdot \text{m}^2), \quad (5.19)
\]

where \( \hat{n} \) is the surface normal of the loop and governed by the following right-hand rule: when the four fingers of the right hand advance in the direction of the current \( I \), the direction of the thumb specifies the direction of \( \hat{n} \). In terms of \( m \), the torque vector \( \mathbf{T} \) can be written as

\[
\mathbf{T} = m \times \mathbf{B} \quad (\text{N} \cdot \text{m}). \quad (5.20)
\]

Figure 5-7: Rectangular loop in a uniform magnetic field with flux density \( B \) whose direction is perpendicular to the rotation axis of the loop, but makes an angle \( \theta \) with the loop’s surface normal \( \hat{n} \).
Jean-Baptiste Biot

Born: 21 April 1774, Paris
Died: 3 February 1862 (aged 87), Paris
Nationality: French
Fields: Physics, astronomy and mathematics
Known for: Biot-Savart law
Influenced: Louis Pasteur, William Ritchie

Signature

Jean-Baptiste Biot

Félix Savart

Félix Savart was the son of Gérard Savart, an engineer at the military school of Metz. His brother, Nicolas, student at École Polytechnique and officer in the engineering corps, did work on vibration. Wikipedia.

Born: June 30, 1791, Charleville-Mézières
Died: March 16, 1841, Paris
Biot-Savart Law

Magnetic field induced by a differential current:

\[ dH = \frac{I}{4\pi} \frac{dl \times \hat{R}}{R^2} \quad \text{(A/m)} \]

For the entire length:

\[ H = \frac{I}{4\pi} \int_{l} \frac{dl \times \hat{R}}{R^2} \quad \text{(A/m),} \quad (5.22) \]

where \( l \) is the line path along which \( I \) exists.

Figure 5-8: Magnetic field \( dH \) generated by a current element \( I \, dl \). The direction of the field induced at point \( P \) is opposite to that induced at point \( P' \).
Magnetic Field due to Current Densities

\[ H = \frac{1}{4\pi} \int_S \frac{J_s \times \hat{R}}{R^2} \, ds \quad \text{(surface current)}, \]

\[ H = \frac{1}{4\pi} \int_V \frac{J \times \hat{R}}{R^2} \, dV \quad \text{(volume current)}. \]

Figure 5-9: (a) The total current crossing the cross section \( S \) of the cylinder is \( I = \int_S J \cdot ds \). (b) The total current flowing across the surface of the conductor is \( I = \int_l J_s \, dl \).
Example 5-2: Magnetic Field of Linear Conductor

**Solution:** From Fig. 5-10, the differential length vector \( dl = \hat{z} \, dz \). Hence, \( dl \times \hat{R} = dz \, (\hat{z} \times \hat{R}) = \hat{\phi} \sin \theta \, dz \), where \( \hat{\phi} \) is the azimuth direction and \( \theta \) is the angle between \( dl \) and \( \hat{R} \). Application of Eq. (5.22) gives

\[
H = \frac{I}{4\pi} \int_{z=-l/2}^{z=l/2} \frac{dl \times \hat{R}}{R^2} = \hat{\phi} \frac{I}{4\pi} \int_{-l/2}^{l/2} \frac{\sin \theta}{R^2} \, dz. \tag{5.25}
\]

Both \( R \) and \( \theta \) are dependent on the integration variable \( z \), but the radial distance \( r \) is not. For convenience, we will convert the integration variable from \( z \) to \( \theta \) by using the transformations

\[
R = r \csc \theta, \tag{5.26a}
\]

\[
z = -r \cot \theta, \tag{5.26b}
\]

\[
dz = r \csc^2 \theta \, d\theta. \tag{5.26c}
\]

**Figure 5-10:** Linear conductor of length \( l \) carrying a current \( I \). (a) The field \( dH \) at point \( P \) due to incremental current element \( dl \). (b) Limiting angles \( \theta_1 \) and \( \theta_2 \), each measured between vector \( I \, dl \) and the vector connecting the end of the conductor associated with that angle to point \( P \) (Example 5-2).
Upon inserting Eqs. (5.26a) and (5.26c) into Eq. (5.25), we have

\[ H = \frac{\hat{\phi}}{4\pi} \int_{\theta_1}^{\theta_2} \frac{\sin \theta \ r \ \text{csc}^2 \ \theta \ d\theta}{r^2 \ \text{csc}^2 \ \theta} \]

\[ = \frac{\hat{\phi}}{4\pi r} \int_{\theta_1}^{\theta_2} \sin \theta \ d\theta \]

\[ = \frac{\hat{\phi}}{4\pi r} (\cos \theta_1 - \cos \theta_2), \quad (5.27) \]

where \( \theta_1 \) and \( \theta_2 \) are the limiting angles at \( z = -l/2 \) and \( z = l/2 \), respectively. From the right triangle in Fig. 5-10(b), it follows that

\[ \cos \theta_1 = \frac{l/2}{\sqrt{r^2 + (l/2)^2}}, \quad (5.28a) \]

\[ \cos \theta_2 = -\cos \theta_1 = \frac{-l/2}{\sqrt{r^2 + (l/2)^2}}. \quad (5.28b) \]

Hence,

\[ B = \mu_0 H = \frac{\mu_0 H}{2\pi r \sqrt{4r^2 + l^2}} \quad \text{(T).} \quad (5.29) \]

For an infinitely long wire with \( l \gg r \), Eq. (5.29) reduces to

\[ B = \frac{\mu_0 I}{2\pi r} \quad \text{(infinitely long wire).} \quad (5.30) \]

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**Example 5-2: Magnetic Field of Linear Conductor**

![Diagram](Figure 5-10: Linear conductor of length \( l \) carrying a current \( I \). (a) The field \( dH \) at point \( P \) due to incremental current element \( dl \). (b) Limiting angles \( \theta_1 \) and \( \theta_2 \), each measured between vector \( I \) \( dl \) and the vector connecting the end of the conductor associated with that angle to point \( P \) (Example 5-2).)
Magnetic Field of a Current Carrying Inductor

2nd method of calculations using Biot-Savart Law.

\[ dl = dz \hat{z}, \]
\[ R^2 = r^2 + z^2, \]
\[ dl \times \hat{R} = d\zeta \left( r^2 \hat{r} + z \hat{z} \right) = 0, \]
\[ \frac{\hat{z} \times \hat{r}}{r^2} = \hat{\varphi}. \]

\[ \Phi \Rightarrow \oint \vec{B} \cdot d\vec{l} \]
\[ dl \times \hat{R} = \frac{dl \times \vec{R}}{l^2}, \]
\[ \vec{B} \sim \frac{l}{4\pi} \frac{dl \times \vec{R}}{l^2}. \]

\[ \vec{B}_z = \frac{l}{2\pi} \frac{dl}{\sqrt{r^2 + z^2}} \]
\[ \Rightarrow \frac{l}{4\pi} \frac{dl}{\sqrt{r^2 + z^2}} = \frac{l}{2\pi} \frac{dz}{\sqrt{r^2 + z^2}} \times \frac{dl}{\sqrt{r^2 + z^2}}, \]
\[ \Rightarrow \vec{B} \rightarrow \frac{l}{2\pi} \frac{dl}{\sqrt{r^2 + z^2}} \]
\[ \Rightarrow \frac{l}{2\pi} \frac{dl}{\sqrt{r^2 + z^2}} \Rightarrow as \ l \rightarrow \infty. \]
Magnetic Field of Long Conductor

\[ B = \hat{\Phi} \frac{\mu_0 I}{2\pi r} \]  
(infinitely long wire).
**CD Module 5.2 Magnetic Fields due to Line Sources**

You can place $z$-directed linear currents anywhere in the display plane ($x$-$y$ plane), select their magnitudes and directions, and then observe the spatial pattern of the induced magnetic flux $B(x, y)$. 

[Image of a module interface with magnetic field visualization and input options for line sources and magnetic field calculations.]
Let's now do an example for test (5,5)
we want field at point O

To consider the situation:

1. Biot-Savart law: tells us that magnetic fields circulate around wires, so at O,
   no contribution from section of the horizontal length.

2. For the st radical moment $\overrightarrow{OA}, \overrightarrow{OC}$, let
   $\overrightarrow{dL} = \overrightarrow{OC} - \overrightarrow{OA}$,
   $\overrightarrow{dL} \times \overrightarrow{R} = 0$,
   $\overrightarrow{dL} \times \overrightarrow{R} = 0$

3. All that is left is the AC, where $\overrightarrow{dL} \perp \overrightarrow{R}$,
   $\overrightarrow{dL} \times \overrightarrow{R} = 0$, $\overrightarrow{dL} \times \overrightarrow{R} = 0$

   0 = $(dL)^2$

   $\overrightarrow{H} = \int d\overrightarrow{H} = \int \frac{I}{2\pi} \frac{dL^2}{a^2} = \frac{\mu_0 I}{2\pi a^2}$

   $\overrightarrow{H} = \frac{\mu_0 I}{2\pi a^2} \overrightarrow{R}$

If we have a complete circular loop $\alpha \rightarrow 2\pi \rightarrow \overrightarrow{H} \rightarrow \frac{\mu_0 I}{2\pi} \overrightarrow{R}$.
EE 135, Winter 2013

Reading: Chapter 4. chapter 4.10-4.11, chapter 5.1-5.4

Homework #4, due 2/14: Chapter 4, problems, 4.10, 4.21, 4.24, 4.31, 4.46

For laboratory: read lab. introduction before lab.

NOTE: Midterm# 2 is on February 28, not February 21.

Lecture 10