EE 135, Winter 2013

Reading: Chapter 7

Homework #6: problems, 6.6, 6.9, 6.18, 6.22, 6.27

Due March 12, 2013, today

Homework #7: problems, 7.6, 7.11, 7.26, 7.32
Due March 14, this Thursday

FINAL EXAM: March 22, 12-3pm/next Friday

Lecture 17
Attenuation

Magnitude of $E$

$$|\vec{E}_x(z)| = |E_{x0}e^{-\alpha z}e^{-j\beta z}| = |E_{x0}|e^{-\alpha z}$$

Skin depth

$$\delta_s = \frac{1}{\alpha} \text{ (m),} \quad (7.72)$$

Figure 7-13: Attenuation of the magnitude of $\vec{E}_x(z)$ with distance $z$. The skin depth $\delta_s$ is the value of $z$ at which $|\vec{E}_x(z)|/|E_{x0}| = e^{-1}$, or $z = \delta_s = 1/\alpha$.

the wave magnitude decreases by a factor of $e^{-1} \approx 0.37$ (Fig. 7-13). At depth $z = 3\delta_s$, the field magnitude is less than 5% of its initial value, and at $z = 5\delta_s$, it is less than 1%.

This distance $\delta_s$, called the **skin depth** of the medium, characterizes how deep an electromagnetic wave can penetrate into a conducting medium.
Low and High Frequency Approximations

Table 7-1: Expressions for $\alpha$, $\beta$, $\eta_c$, $u_p$, and $\lambda$ for various types of media.

<table>
<thead>
<tr>
<th></th>
<th>Any Medium</th>
<th>Lossless Medium ($\sigma = 0$)</th>
<th>Low-loss Medium ($\varepsilon''/\varepsilon' \ll 1$)</th>
<th>Good Conductor ($\varepsilon''/\varepsilon' \gg 1$)</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = \omega \left[ \frac{\mu \varepsilon'}{2} \left( \sqrt{1 + \left( \frac{\varepsilon''}{\varepsilon'} \right)^2} - 1 \right) \right]^{1/2}$</td>
<td></td>
<td>0</td>
<td>$\frac{\sigma}{2\sqrt{\varepsilon}}$</td>
<td>$\sqrt{\pi f \mu \sigma}$</td>
<td>(Np/m)</td>
</tr>
<tr>
<td>$\beta = \omega \left[ \frac{\mu \varepsilon'}{2} \left( \sqrt{1 + \left( \frac{\varepsilon''}{\varepsilon'} \right)^2} + 1 \right) \right]^{1/2}$</td>
<td></td>
<td>$\omega \sqrt{\mu \varepsilon}$</td>
<td>$\omega \sqrt{\mu \varepsilon}$</td>
<td>$\sqrt{\pi f \mu \sigma}$</td>
<td>(rad/m)</td>
</tr>
<tr>
<td>$\eta_c = \frac{\sqrt{\mu}}{\sqrt{\varepsilon}} \left( 1 - j \frac{\varepsilon''}{\varepsilon'} \right)^{-1/2}$</td>
<td>$\frac{\sqrt{\mu}}{\sqrt{\varepsilon}}$</td>
<td>$\frac{\sqrt{\mu}}{\sqrt{\varepsilon}}$</td>
<td>$\frac{\sqrt{\mu}}{\sqrt{\varepsilon}}$</td>
<td>$(1 + j) \frac{\alpha}{\sigma}$</td>
<td>(Ω)</td>
</tr>
<tr>
<td>$u_p = \frac{\omega}{\beta}$</td>
<td>$1/\sqrt{\mu \varepsilon}$</td>
<td>$1/\sqrt{\mu \varepsilon}$</td>
<td>$1/\sqrt{4\pi f / \mu \sigma}$</td>
<td>$u_p / f$</td>
<td>(m/s)</td>
</tr>
<tr>
<td>$\lambda = \frac{2\pi}{\beta} = \frac{u_p}{f}$</td>
<td>$\frac{u_p}{f}$</td>
<td>$\frac{u_p}{f}$</td>
<td>$\frac{u_p}{f}$</td>
<td>$\frac{u_p}{f}$</td>
<td>(m)</td>
</tr>
</tbody>
</table>

Notes: $\varepsilon' = \varepsilon$; $\varepsilon'' = \sigma / \omega$; in free space, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$; in practice, a material is considered a low-loss medium if $\varepsilon''/\varepsilon' = \sigma / \omega \varepsilon < 0.01$ and a good conducting medium if $\varepsilon''/\varepsilon' > 100$. 
dc vs ac Current Flow in Conductors

![Diagram](image)

(a) dc case

(b) ac case

Figure 7-14: Current density $\mathbf{J}$ in a conducting wire is (a) uniform across its cross section in the dc case, but (b) in the ac case, $\mathbf{J}$ is highest along the wire’s perimeter.
AC Current Flow

dielectric

Conductor

propagation direction.

x-polarized E field, \( \hat{E} = \hat{x} E_0 \) in dielectric because of boundary condition, \( E_{TBX} \) continuous.

\[ \hat{E}(z) = \hat{x} E_0 e^{-\alpha z} e^{-j\beta z} \]

\[ \hat{H}(z) = \frac{j E_0}{\eta_0} e^{-\alpha z} e^{-j\beta z} \]

and \( J = \sigma \hat{E} \rightarrow \hat{J}(z) = \hat{x} \hat{J}_x(z) \)

which is attenuated as we go into conductor.
Linear Conductor

For a conductor with \( E_0 \) at the surface:

\[
\begin{align*}
\vec{E}(z) &= \hat{x} E_0 e^{-\alpha z} e^{-j\beta z}, \\
\vec{H}(z) &= \hat{y} \frac{E_0}{\eta_c} e^{-\alpha z} e^{-j\beta z}.
\end{align*}
\] (7.83a, 7.83b)

From \( \vec{J} = \sigma \vec{E} \), the current flows in the \( x \)-direction, and its density is

\[
\vec{J}(z) = \hat{x} \vec{J}_x(z),
\] (7.84)

with

\[
\vec{J}_x(z) = \sigma E_0 e^{-\alpha z} e^{-j\beta z} = J_0 e^{-\alpha z} e^{-j\beta z},
\] (7.85)

Total current crossing \( y-z \) plane:

\[
\begin{align*}
\vec{I} &= w \int_{0}^{\infty} \vec{J}_x(z) \, dz \\
&= w \int_{0}^{\infty} J_0 e^{-(1+j)z/\delta_s} \, dz = \frac{J_0 w \delta_s}{(1+j)}. \quad (A)
\end{align*}
\]

Figure 7-15: Exponential decay of current density \( \vec{J}_x(z) \) with \( z \) in a solid conductor. The total current flowing through (a) a section of width \( w \) extending between \( z = 0 \) and \( z = \infty \) is equivalent to (b) a constant current density \( J_0 \) flowing through a section of depth \( \delta_s \).
# Low and High Frequency Approximations

Table 7-1: Expressions for $\alpha$, $\beta$, $\eta_c$, $u_p$, and $\lambda$ for various types of media.

<table>
<thead>
<tr>
<th></th>
<th>Any Medium</th>
<th>Lossless Medium ($\sigma = 0$)</th>
<th>Low-loss Medium ($\varepsilon'' / \varepsilon' \ll 1$)</th>
<th>Good Conductor ($\varepsilon'' / \varepsilon' \gg 1$)</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$\omega \left[ \frac{\mu \varepsilon'}{2} \left( \sqrt{1 + \left( \frac{\varepsilon''}{\varepsilon'} \right)^2} - 1 \right) \right]^{1/2}$</td>
<td>0</td>
<td>$\frac{\sigma}{2} \sqrt{\frac{\mu}{\varepsilon}}$</td>
<td>$\sqrt{\pi f \mu \sigma}$</td>
<td>(Np/m)</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$\omega \left[ \frac{\mu \varepsilon'}{2} \left( \sqrt{1 + \left( \frac{\varepsilon''}{\varepsilon'} \right)^2} + 1 \right) \right]^{1/2}$</td>
<td>$\omega \sqrt{\mu \varepsilon}$</td>
<td>$\omega \sqrt{\mu \varepsilon}$</td>
<td>$\sqrt{\pi f \mu \sigma}$</td>
<td>(rad/m)</td>
</tr>
<tr>
<td>$\eta_c$</td>
<td>$\sqrt{\frac{\mu}{\varepsilon'}} \left( 1 - j \frac{\varepsilon''}{\varepsilon'} \right)^{-1/2}$</td>
<td>$\sqrt{\frac{\mu}{\varepsilon}}$</td>
<td>$\sqrt{\frac{\mu}{\varepsilon}}$</td>
<td>$(1 + j) \frac{\alpha}{\sigma}$</td>
<td>(Ω)</td>
</tr>
<tr>
<td>$u_p$</td>
<td>$\frac{\omega}{\beta}$</td>
<td>$1/\sqrt{\mu \varepsilon}$</td>
<td>$1/\sqrt{\mu \varepsilon}$</td>
<td>$\sqrt{4 \pi f / \mu \sigma}$</td>
<td>(m/s)</td>
</tr>
<tr>
<td>$\lambda = \frac{2 \pi / \beta = u_p / f}{u_p / f}$</td>
<td></td>
<td>$1/\sqrt{\mu \varepsilon}$</td>
<td>$1/\sqrt{\mu \varepsilon}$</td>
<td>$u_p / f$</td>
<td>(m)</td>
</tr>
</tbody>
</table>

Notes: $\varepsilon' = \varepsilon$; $\varepsilon'' = \sigma / \omega$; in free space, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$; in practice, a material is considered a low-loss medium if $\varepsilon'' / \varepsilon' = \sigma / \omega \varepsilon < 0.01$ and a good conducting medium if $\varepsilon'' / \varepsilon' > 100$. 
Surface Impedance

The voltage across a length \( l \) at the surface [Fig. 7-15(b)] is given by

\[
\tilde{V} = E_0 l = \frac{J_0}{\sigma} l. \tag{7.88}
\]

Hence, the impedance of a slab of width \( w \), length \( l \), and depth \( d = \infty \) (or, in practice, \( d > 5\delta_s \)) is

\[
Z = \frac{\tilde{V}}{I} = \frac{1 + j}{\sigma \delta_s} \frac{l}{w} \quad (\Omega). \tag{7.89}
\]

It is customary to represent \( Z \) as

\[
Z = Z_s \frac{l}{w}, \tag{7.90}
\]

where \( Z_s \), the internal or surface impedance of the conductor, is defined as the impedance \( Z \) for a length \( l = 1 \) m and a width \( w = 1 \) m. Thus,

\[
Z_s = \frac{1 + j}{\sigma \delta_s} \quad (\Omega). \tag{7.91}
\]

Thus, conductor is equivalent to a resistor in series with an inductor.

Figure 7-15: Exponential decay of current density \( \tilde{J}_x(z) \) with \( z \) in a solid conductor. The total current flowing through (a) a section of width \( w \) extending between \( z = 0 \) and \( z = \infty \) is equivalent to (b) a constant current density \( J_0 \) flowing through a section of depth \( \delta_s \).
Surface Impedance
(good conductor)

\[ Z_s = \frac{1 + j}{\sigma s} = \frac{1}{\sigma s} + \frac{j}{\sigma s} \]

\[ \text{Resistance} \quad \text{Inductance} \]

\[ \text{Frequency} \]

\[ Z_s = R_s + j\pi L_s \implies \pi L_s = \frac{1}{\sigma s} \]

\[ R_s = \frac{1}{\sigma s} = \sqrt{\frac{H m}{\sigma}} \]

\[ L_s = \frac{1}{2\pi f} \sigma s \]

\[ L_s = \frac{1}{2} \frac{M}{\pi f s} \]

\[ \vdots \]

AC resistance of a slab of width \( W \), length \( l \) m:

\[ R_{AC} = \frac{R_s}{W} = \frac{l}{\sigma s W} \]

\[ \text{equiv to } R_{AC} = \frac{1}{\sigma A} \]

\[ \text{when } A = \sigma s W (\text{section}) \]
ac Resistance of Coaxial Cable

Since in the ac case, most of the current flows through a very thin skin along the outside of the inner conductor and along the inside of the outer conductor, we can use the results of the planar conductor to figure out the resistance of the coax. The procedure leads to the following expression for the resistance per unit length:

\[ R_s = \left( \frac{\pi \mu}{\sigma} \right)^{1/2} \]

\[ R' = R'_1 + R'_2 = \frac{R_s}{2\pi} \left( \frac{1}{a} + \frac{1}{b} \right) \] (Ω/m)

**Figure 7-16:** The inner conductor of the coaxial cable in (a) is represented in (b) by a planar conductor of width \( 2\pi a \) and depth \( \delta_s \), as if its skin has been cut along its length on the bottom side and then unfurled into a planar geometry.
Power dissipation in Microwave Oven

Assume material has conductivity, $\sigma$

$\vec{E}$ is propagating in $\hat{z}$ direction and

$\vec{E} = \hat{z} E_0 \cos \omega t$

$\vec{E}$ field creates a potential difference across the faces ($\perp \hat{y}$ direction). They have an interaction area $= A$

Potential diff. $V = -\int \vec{E} \cdot d\vec{l}$

$= -Ed$
Power dissipation in Microwave Oven

Assume material has conductivity, \( \sigma \)

\[ \vec{E} \text{ is propagating in } \hat{z} \text{ direction and} \]

\[ \vec{E} = \hat{z} E_0 \text{ const} \]

\( \vec{E} \) field creates a potential difference across the faces (\( \perp \hat{y} \) direction).

They have cross section area = \( A \)

Potential diff. \( V = -\int \vec{E} \cdot d\vec{s} \)

\[ = -Ed \]

The current that the \( V \) produces is \( I = \int \vec{J} \cdot d\vec{s} = \sigma E A \)

\[ = \frac{Ed}{J} \]

\( \therefore \) Power produced in this volume is:

\[ P = IV = (\sigma EA)(Ed) = \sigma E^2 Ad \text{ per volume} \]

\[ \therefore \text{ power/ volume} = \sigma E^2 \]

Assume microwave oven operates at \( f = 2 \text{GHz} \)

And produce a peak \( E \) of \( E_0 = 4 \times 10^4 \text{ V/m} \)
Power Dissipation in Microwave Oven

\[ \text{time averaged Power dissipated/volume} \]

\[
\frac{P_{\text{avg}}}{\text{vol}} = \frac{1}{T} \int_0^T \sigma E_0^2 \omega^2 \omega t \, dt
\]

\[
= \sigma E_0^2 \left[ \frac{1}{T} \int_0^T \omega^2 \omega t \, dt \right]
\]

\[
= \frac{1}{2} \text{ indep. of freq.}
\]

\[
\therefore \frac{P_{\text{avg}}}{\text{vol}} = \frac{1}{2} \sigma E_0^2
\]

For meat, \( \sigma \approx 1 \text{ Siemans/m at 2 GHz} \)

\[
\frac{P_{\text{avg}}}{\text{vol}} = \frac{1}{2} \times 1 \times (4 \times 10^{-4})^2 \, \frac{\text{watts}}{\text{m}^3}
\]

\[
= 8 \times 10^8 \, \frac{\text{watts}}{\text{m}^3} \rightarrow 0.8 \, \frac{\text{watts}}{\text{mm}^3}
\]
Figure 4. Visualization of the horizontal mode structure in a microwave oven using infrared thermal imaging. A glass plate with a thin water film was placed at a height of 8 cm and heated for 15 s with a microwave power of 800 W without using the turntable (for more details on the experiment, see [10]).
Figure 6. Absorption coefficient for water from microwaves to the UV (after [7]).

7. Dielectric constant of food

Figure 7. Dielectric constants of various kinds of food (after [13]).

Power Flow in Transmission Lines

\[ v(\theta, t) = \text{Re} \left( \tilde{V} e^{j\omega t} \right) = \text{Re} \left( V_0 e^{j\phi} e^{j(\omega t + \theta)} \right) = V_0 \cos(\theta + \Delta \theta) \]

Let's assume \( \Delta = 0 \) for now - make it simple.

Similarly, current \( i(\theta, t) = \frac{V_0}{Z_0} \cos(\theta + \Delta \theta) \)

Instant power \( p(\theta, t) = V_0 \cos(\theta + \Delta \theta) \)

Average power \( P_{\text{avg}} = \frac{1}{T} \int_0^T \text{Re}(p(\theta, t)) \, dt \)

But \( V_0^2 \cos^2(\theta + \Delta \theta) = \frac{1}{2} (V_0^2 + V_0^2 \cos(2(\omega t + \phi))) \)

\[ P_{\text{avg}} = \frac{1}{2T} \int_0^T \frac{V_0^2}{Z_0} \left(1 + \cos(2(\omega t + \phi))\right) \, dt = \frac{1}{2} \frac{V_0^2}{Z_0} = \frac{1}{2} |V_0|^2 = \frac{1}{2} |V_0||I| = \]

\[ P_{\text{avg}} = \frac{1}{2} \text{Re} \left[ \tilde{V} \tilde{I}^* \right] \]

\[ S_{\text{avg}} = \frac{1}{2} \text{Re} \left[ \tilde{E} \times \tilde{H}^* \right] \]

By J. H. POYNTING, M.A., late Fellow of Trinity College, Cambridge, Professor of Physics, Mason College, Birmingham.

Communicated by Lord RAYLEIGH, M.A., D.C.L., F.R.S.

Received December 17, 1883, — Read January 10, 1884.

Philosophical Transactions of the Royal Society of London, Volume 175, pp. 343-361, Online
Power Flow of EM Waves 1.

From Maxwell:
\[ \mathbf{\nabla} \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \]

\[ \mathbf{\nabla} \times \mathbf{H} = \mathbf{j} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \]

\[ \therefore \mathbf{E} \cdot (\mathbf{\nabla} \times \mathbf{H}) = \sigma \mathbf{E} \cdot \mathbf{E} + \varepsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \sigma \mathbf{E}^2 + \frac{1}{2} \varepsilon \frac{\partial}{\partial t} \mathbf{E}^2 \]
\[ \mathbf{E} \cdot (\nabla \times \mathbf{H}) = \sigma \mathbf{E} \cdot \mathbf{E} + \epsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \sigma \mathbf{E}^2 + \frac{1}{2} \epsilon \frac{\partial \mathbf{E}^2}{\partial t} \]

General relation for vector fields, \( \mathbf{\hat{a}}, \mathbf{\hat{b}} \):

\[ \nabla \cdot (\mathbf{\hat{a}} \times \mathbf{\hat{b}}) = \mathbf{\hat{b}} \cdot (\nabla \times \mathbf{\hat{a}}) - \mathbf{\hat{a}} \cdot (\nabla \times \mathbf{\hat{b}}) \]

1. Let \( \mathbf{\hat{a}} = \mathbf{H}, \mathbf{\hat{b}} = \mathbf{\hat{E}} \)

\[ \nabla \cdot (\mathbf{H} \times \mathbf{\hat{E}}) = \mathbf{\hat{E}} \cdot (\nabla \times \mathbf{H}) - \mathbf{H} \cdot (\nabla \times \mathbf{\hat{E}}) \]

\[ \begin{align*}
\therefore \mathbf{\hat{H}} \cdot (\nabla \times \mathbf{\hat{E}}) + \mathbf{\hat{E}} \cdot (\nabla \times \mathbf{H}) &= \mathbf{E} \cdot (\nabla \times \mathbf{H}) \\
&= \sigma \mathbf{E}^2 + \frac{1}{2} \epsilon \frac{\partial \mathbf{E}^2}{\partial t} \end{align*} \]

2. Let \( \mathbf{\hat{a}} = \frac{\partial}{\partial t}, \mathbf{\hat{b}} = \mathbf{H} \)

\[ \nabla \cdot (\frac{\partial}{\partial t} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \frac{\partial}{\partial t}) - \frac{\partial}{\partial t} \cdot (\nabla \times \mathbf{H}) \]

\[ = -\frac{1}{2} \mu \frac{\partial^2}{\partial t^2} (\mathbf{H}^2) - \nabla \cdot (\mathbf{E} \times \mathbf{H}) = \sigma \mathbf{E}^2 + \frac{1}{2} \epsilon \frac{\partial \mathbf{E}^2}{\partial t} \]

\[ \therefore \nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\frac{1}{2} \frac{\partial}{\partial t} \left[ \mu \mathbf{E}^2 + \epsilon \mathbf{E}^2 \right] - \sigma \mathbf{E}^2 \]
Power Flow of EM Waves 2

\[ \nabla \cdot (E \times H) = -\frac{1}{2} \frac{\delta}{\delta t} \left[ \mu H^2 + \epsilon E^2 \right] - \sigma E^2 \]

volume integral of both sides.

\[ \int \nabla \cdot (E \times H) \, dv = -\frac{\delta}{\delta t} \int \left[ \frac{1}{2} (\mu H^2 + \epsilon E^2) \right] \, dv - \int \sigma E^2 \, dv \]

but div. theorem says: \[ \int \nabla \cdot (E \times H) \, dv = \oint \left( E \times H \right) \cdot d\mathbf{s} \]

Poynting’s theorem

\[ \oint \left( E \times H \right) \cdot d\mathbf{s} = -\frac{\delta}{\delta t} \int \left[ \frac{1}{2} (\mu H^2 + \epsilon E^2) \right] \, dv - \int \sigma E^2 \, dv \]

total power leaving the volume defined by closed surface \( S \)

rate of decrease of the energy stored in the \( E, H \) fields

Ohmic power dissipated

define \( \vec{S} = E \times H \), the “Poynting” vector points in direction of propagation.

Power, \( P = \int_S \vec{S} \cdot d\mathbf{s} = \int_S \vec{S} \cdot n \, dA \)
Poynting’s Theorem
Power Density

Poynting vector:
\[ S = E \times H \quad (W/m^2). \]

Total power intercepted by A:
\[ P = \int_A S \cdot \hat{n} dA \quad (W). \]

Time-average power density:
\[ S_{av} = \frac{1}{2} \Re \left[ \widetilde{E} \times \widetilde{H}^* \right] \quad (W/m^2). \]
Power Density Carried by Plane Wave

For a plane wave with $E$ field:

$$\vec{E}(z) = \hat{x} \vec{E}_x(z) + \hat{y} \vec{E}_y(z)$$

$$= (\hat{x} E_{x0} + \hat{y} E_{y0})e^{-jkz},$$

the average power density carried by the wave is:

$$S_{av} = \hat{z} \frac{1}{2\eta} (|E_{x0}|^2 + |E_{y0}|^2)$$

$$\quad = \hat{z} \frac{|\vec{E}|^2}{2\eta} \quad (\text{W/m}^2),$$
Power Flow in Transmission Lines

Consider just left \( \rightarrow \) right going waves

\[
\text{voltage } v(z,t) = \text{Re} \left[ \tilde{v} e^{j\omega t} \right] \\
= \text{Re} \left[ |V_0| e^{j\phi} e^{j(\beta z + \omega t)} \right] \\
= |V_0| u_0 \cos(\omega t + \beta z + \phi)
\]

lets assume \( \phi = 0 \) for now - make it simple

Similarly, current \( i(t) = \frac{|V_0| u_0}{Z_0} \cos(\omega t + \beta z) \)

\[
\text{instant power } p(t) = v(t)i(t) \\
= \frac{|V_0|^2}{Z_0} u_0^2 \cos^2(\omega t + \beta z)
\]

\[
\text{avg power } P_{\text{avg}} = \frac{1}{T} \int_0^T p(t) \, dt
\]

\[
\text{but } u_0^2 \cos^2(\omega t + \beta z) = \frac{1}{2} (1 + \cos[2(\omega t + \beta z)])
\]

\[
\therefore P_{\text{avg}} = \frac{1}{2T} \int_0^T \left( \frac{|V_0|^2}{Z_0} \cos[2(\omega t + \beta z)] \right) \, dt
\]

\[
= \frac{1}{2} \left( \frac{|V_0|^2}{Z_0} \right) \int_0^T 1 \, dt
\]

\[
P_{\text{avg}} = \frac{1}{2} \text{Re} \left[ \tilde{V} \tilde{I}^* \right]
\]

\[
S_{\text{avg}} = \frac{1}{2} \text{Re} \left[ \tilde{E} \tilde{H}^* \right]
\]
\[\tilde{E}(z) = \chi E_x(z) + \tilde{g} E_y(z)\]
\[= (\chi E_{x0} + \tilde{g} E_{y0}) e^{-j\kappa z}\]

[\hat{H}(z) = \frac{1}{\eta} \hat{R} x \hat{E} = \frac{1}{\eta} \hat{z} x \hat{E}\]

\[\therefore \hat{H}(z) = \frac{1}{\eta} \left[ \hat{z} x (\chi E_{x0} + \tilde{g} E_{y0}) e^{-j\kappa z} \right]\]
\[= \frac{1}{\eta} \left[ \tilde{g} E_{x0} - \chi E_{y0} \right] e^{-j\kappa z}\]

Then \[\tilde{E} \times H^* = (\chi E_{x0} + \tilde{g} E_{y0}) x \frac{1}{\eta} \left[ \tilde{g} E_{x0} - \chi E_{y0} \right]\]
\[= \frac{1}{\eta} \left( \chi E_{x0}^2 + \tilde{g} E_{y0}^2 \right) \hat{z}\]

[\tilde{S}_{avg} = \hat{z} \frac{1}{2\eta} |\tilde{E}|^2]
Solar Power

If illumination of solar irradiation characterized by a power density of 1 kW/m²:

A) what is total power radiated by Sun
B) total solar power intercepted by earth
C) $E^*$ of the solar irradiation at earth's surface assuming all illumination at single frequency.

$R_{\text{Earth Orbit}} \approx 1.5 \times 10^8 \text{ km} = R_S$

$R_{\text{Earth}} \approx 6.38 \times 10^3 \text{ km} = R_E$

assuming solar irradiation is isotropic.

A) total power radiated
Solar Power

If illumination of solar irradiation is characterized by a power density of 1 kW/m²:

A) what is total power radiated by sun
B) total solar power intercepted by earth
C) $E_0$ of the solar irradiation at earth's surface assuming all illumination at single frequency.

$R_{\text{Earth Orbit}} \approx 1.5 \times 10^8$ K$m = R_S$

$R_{\text{Earth}} \approx 6.38 \times 10^3$ K$m = R_E$

Assuming solar irradiation is isotropic.

A) Total power radiated

$$P_{\text{Sun}} = S_{\text{ANG}} \times (4\Pi R_S^2), \quad \text{where } S_{\text{ANG}} = \frac{1}{2} R_e \left[ \frac{E_0}{4\Pi} \right]$$

and

$$P = \int_S n dA$$

power density

$$S_{\text{ANG}} = 10^3 \text{ W/m}^2$$

$$P_{\text{Sun}} = 10^3 \text{ W/m}^2 \times (4\Pi \times [1.5 \times 10^8 \text{ m}])^2$$

$$P_{\text{Sun}} \approx 2.3 \times 10^{26} \text{ Watts}$$
Solar Power (2)

B. What is total solar power intercepted by Earth?

Power intercepted by Earth cross-section:

\[ P_{\text{INT}} = S_{\text{AXF}} \left( \frac{TR_{E}^2}{T} \right) \]

**Note:** Not surface area of Earth

\[ P_{\text{INT}} = 4 \times 10^3 \text{ watts/m}^2 \]

\[ P_{\text{INT}} = (6.4 \times 10^6 \text{ m})^2 \]

\[ P_{\text{INT}} = 1.28 \times 10^{11} \text{ watts} \]

**Note:** \( \frac{P_{\text{INT}}}{P_{\text{SUN}}} = 1.28 \times 10^{11} \)

\[ \frac{1.28 \times 10^{11}}{2.8 \times 10^{26}} = 0.46 \times 10^{-15} \]
C. to get at $E^2$ at earth surface due to solar irradiation

$$S_{\text{avg}} = \frac{1}{2} \text{Re}[\tilde{E} \times \tilde{H}^*]$$

$$S_{\text{avg}} = \frac{1}{2} |\tilde{E}|^2$$

$$E_0 = \sqrt{2} V_0 S_{\text{avg}} = \sqrt{2 \times 377.5 \times 10^3 \text{watt/m}^2}$$

watt = V \cdot A, \ \frac{V}{A} \Rightarrow \sqrt{\text{watt}} = \sqrt{V^2/m^2} \rightarrow V/m$$

$E_0 = 870 \text{ V/m}$
Plane Wave in Lossy Media

\[ \mathbf{E}(\mathbf{z}) = \hat{x} E_x(\mathbf{z}) + \hat{y} E_y(\mathbf{z}) \]
\[ = (\hat{x} E_{x0} + \hat{y} E_{y0}) e^{-\alpha z} e^{-j\beta z} \]

\[ \mathbf{H}(\mathbf{z}) = \frac{1}{\eta_c} (-\hat{x} E_{y0} + \hat{x} E_{x0}) e^{-\omega z} e^{-j\beta z} \]

where \( \eta_c = \eta_0 e^{j\theta} \)

\[ \mathbf{S}_{\text{mv}}(\mathbf{t}) = \frac{1}{2} \Re\left( \mathbf{E}^* \times \mathbf{H} \right) \]
\[ = \frac{1}{2} \Re\left( \frac{2}{\eta_c} \left( 1|E_{x0}|^2 + |E_{y0}|^2 \right) e^{-2\omega z} \right) \]
\[ = \frac{1}{2} \Re\left( \frac{2}{\eta_0} \left( 1|\mathbf{E}(\omega)|^2 e^{-2\omega z} \right) \right) \]

\[ \mathbf{S}_{\text{mv}}(\mathbf{z}) = \frac{2}{\eta_0} \left( \frac{|\mathbf{E}(\omega)|^2 e^{-2\omega z}}{1|\mathbf{E}(\omega)|^2} \right) \frac{\text{watt}}{m^2} \]

where \( |\mathbf{E}(\omega)|^2 = |E_{x0}|^2 + |E_{y0}|^2 \)

avg power \( \mathbf{S}_{\text{mv}} \) decay as \( e^{-2\omega z} \) not \( e^{-\alpha z} \)
Plane Wave in Lossy Medium

For a plane wave travelling in a lossy medium:

\[
\tilde{\mathbf{E}}(z) = \hat{x} \tilde{E}_x(z) + \hat{y} \tilde{E}_y(z) \\
= (\hat{x} E_{x0} + \hat{y} E_{y0}) e^{-\alpha z} e^{-j\beta z},
\]

\[
\tilde{\mathbf{H}}(z) = \frac{1}{\eta_c} (\hat{x} E_{y0} + \hat{y} E_{x0}) e^{-\alpha z} e^{-j\beta z},
\]

the power density is:

\[
S_{av}(z) = \frac{1}{2} \Re \left[ \tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* \right] \\
= \hat{z} \left( |E_{x0}|^2 + |E_{y0}|^2 \right) e^{-2\alpha z} \Re \left( \frac{1}{\eta_c^*} \right).
\]

By expressing \( \eta_c \) in polar form as

\[
\eta_c = |\eta_c| e^{j\theta_n},
\]

\[
S_{av}(z) = \hat{z} \frac{|\tilde{E}(0)|^2}{2|\eta_c|} e^{-2\alpha z} \cos \theta_n \text{ (W/m}^2\text{)}
\]

Whereas the fields \( \tilde{\mathbf{E}}(z) \) and \( \tilde{\mathbf{H}}(z) \) decay with \( z \) as \( e^{-\alpha z} \), the power density \( S_{av} \) decreases as \( e^{-2\alpha z} \).
Decibel Scale

Gain = G = \frac{P_1}{P_2} \quad \text{power ratio}

G(dB) = 10 \log G = 10 \log \left( \frac{P_1}{P_2} \right)

It provides easier way to look at gains that vary over orders of magnitude.

Note also that if we talk about voltage or E or H gains, the conversion is 20 rather than 10 since power \propto V^2 or E^2 etc. \text{}/

So for power gain G = 10^X, G(dB) = 10XdB
<table>
<thead>
<tr>
<th>G = P_1 / P_2</th>
<th>G (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^x</td>
<td>10X dB</td>
</tr>
<tr>
<td>4</td>
<td>6 dB</td>
</tr>
<tr>
<td>2</td>
<td>3 dB</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>-3</td>
</tr>
<tr>
<td>0.25</td>
<td>-6</td>
</tr>
<tr>
<td>0.10</td>
<td>-10</td>
</tr>
<tr>
<td>10^{-3}</td>
<td>-30dB</td>
</tr>
</tbody>
</table>
Attenuation Rate, \( A \)

the rate of decrease of magnitude of \( S_{av} \) (power density) as a function of distance.

\[
A = 10 \log \left( \frac{S_{av}(z)}{S_{av}(0)} \right) = 10 \log \left( e^{-2a_0 z} \right)
\]

< Remember power density decay as \( e^{-2a_0 z} \) whereas \( E, H \) decay as \( e^{-a_0 z} \) >

\[
\therefore A = -20a_0 z \log(e) \\
= -8.68a_0 z \\
A = -a_0 [\text{dB/m}] z \text{ where } a_0 [\text{dB/m}] = 8.68 a_0 \left( \frac{1}{\text{m}} \right)
\]

and since \( S_{av} \propto |E|^2 \), then

\[
A = 10 \log \left( \frac{|E(z)|^2}{|E(0)|^2} \right) = 20 \log \left( \frac{|E(z)|}{|E(0)|} \right)
\]
Example 7-6: Power Received by a Submarine Antenna

A submarine at a depth of 200 m below the sea surface uses a wire antenna to receive signal transmissions at 1 kHz. Determine the power density incident upon the submarine antenna due to the EM wave of Example 7-4.

Result from Example 7.4:

\[
\begin{align*}
\vec{E}(z,t) &= 3.4 \times 10^4 e^{-12.6z} \cos(2\pi 10^3 t - 0.12z + 60^\circ), \text{ mV/m} \\
\vec{H}(z,t) &= 9 \times 10^3 e^{-12.6z} \cos(2\pi 10^3 t - 0.12z + 15^\circ), \text{ mA/m}
\end{align*}
\]
Example 7-6: Power Received by a Submarine Antenna

A submarine at a depth of 200 m below the sea surface uses a wire antenna to receive signal transmissions at 1 kHz. Determine the power density incident upon the submarine antenna due to the EM wave of Example 7-4.

Result from Example 7.4:

\[ E(t, t) = 2.0 \cdot 10^{-12} \text{ mV} \cdot \text{m}^{-1} (2 \pi \times 10^3 t + 0.1262 + 60^\circ), \text{ mV/m} \]

\[ H(t, t) = 9 \cdot 10^{-12} \text{ mA} \cdot \text{m}^{-1} (2 \pi \times 10^3 t - 0.1262 + 15^\circ), \text{ mA/m} \]

\[ E_0 = 4.4 \text{ mV/m} \]

\[ H_0 = \frac{E_0}{1M \Omega} = 10^2 \text{ mA/m} \]

\[ |\eta_c| = 4.4 \times 10^{-2}, \Omega \]

and \( \theta_\eta = 60 - 15^\circ = 45^\circ \)

\[ \eta_c = 0.044 e^{j45^\circ} \]
Solution: From Example 7-4, \(|\vec{E}(0)| = |E_{x0}| = 4.44\) (mV/m), \(\alpha = 0.126\) (Np/m), and \(\eta_c = 0.044^{\angle45^\circ}\) (Ω). Application of Eq. (7.109) gives

\[
S_{av}(z) = \hat{z} \frac{|E_0|^2}{2|\eta_c|} e^{-2\alpha z} \cos \theta \eta
\]

\[
= \hat{z} \frac{(4.44 \times 10^{-3})^2}{2 \times 0.044} e^{-0.252z} \cos 45^\circ
\]

\[
= \hat{z} 0.16e^{-0.252z} \quad \text{(mW/m}^2\text{)}.
\]

At \(z = 200\) m, the incident power density is

\[
S_{av} = \hat{z} (0.16 \times 10^{-3} e^{-0.252\times200})
\]

\[
= 2.1 \times 10^{-26} \quad \text{(W/m}^2\text{)}.
\]
Problem 7.36  A team of scientists is designing a radar as a probe for measuring the depth of the ice layer over the antarctic land mass. In order to measure a detectable echo due to the reflection by the ice-rock boundary, the thickness of the ice sheet should not exceed three skin depths. If $\varepsilon_r' = 3$ and $\varepsilon_r'' = 10^{-2}$ for ice and if the maximum anticipated ice thickness in the area under exploration is 1.2 km, what frequency range is useable with the radar?
ICE ≤ 3 \delta_{ICE}
\text{skin depth of ice}

\text{since attenuation } \alpha = \frac{1}{\delta_s}

\text{remember that } |E(t)|/|E(0)| = e^{-\alpha z}

\therefore 3 \text{ skin depths results in } e^{-3} \text{ attenuation of } e^{-3} \text{ going in, } e^{-3} \text{ going out!}
\[ t_{\text{MAX}} = 1.2 \times 10^3 \text{ m} \]

\[ t_{\text{MAX}} \leq 3 s \]

\[ 3 s \geq 1.2 \times 10^3 \text{ m} \]

\[ s_s \geq 400 \text{ m/s} \]

\[ \omega \leq \frac{1}{400} = 2.5 \times 10^{-3} / \text{m} \]

\[ \frac{\varepsilon''}{\varepsilon'_r} = \frac{10^{-2}}{3} \ll 1 \text{ from tables } \omega = \frac{\omega}{2} \frac{\sqrt{\mu}}{\varepsilon'} \]

\[ \omega = \frac{2\pi f \varepsilon''}{2} \sqrt{\frac{\mu}{\varepsilon'}} = \frac{\pi f \varepsilon''}{\varepsilon'_r} \sqrt{\frac{\mu_0}{\varepsilon'_r \varepsilon_0}} \]

\[ \omega = \frac{\pi f \varepsilon''}{c} \sqrt{\frac{\mu_0}{\varepsilon'_r}} = \frac{\pi f \varepsilon''}{c \sqrt{\varepsilon'_r}} = \frac{\pi f \times 10^{-2}}{3 \times 10^8 \text{ m/s} \sqrt{3}} \]

\[ \omega = 6 \times 10^{-11} \text{ sec/m} \]

but \( \omega \leq 2.5 \times 10^{-3} / \text{m} \)

\[ \frac{6 \times 10^{-11} \text{ m}}{\text{m}} \leq 2.5 \times 10^{-3} / \text{m} \]

\[ f \leq 41.6 \text{ MHz} \]

NOTE: at \( s_s = \frac{1}{3} t \), the \( \vec{E} \) attenuated \( \frac{1}{3} e^{-\frac{1}{3}} = 0.05 \)

power attenuated \( \text{by} \ e^{-\frac{1}{3}} = 0.0025 \)
Chapter 7 Relationships

**Complex Permittivity**
\[ \varepsilon_c = \varepsilon' - j\varepsilon'' \]
\[ \varepsilon' = \varepsilon \]
\[ \varepsilon'' = \frac{\sigma}{\omega} \]

**Lossless Medium**
\[ k = \omega \sqrt{\mu \varepsilon} \]
\[ \eta = \sqrt{\frac{\mu}{\varepsilon}} \quad (\Omega) \]
\[ u_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu \varepsilon}} \quad (\text{m/s}) \]
\[ \lambda = \frac{2\pi}{k} = \frac{u_p}{f} \quad (\text{m}) \]

**Wave Polarization**
\[ \mathbf{H} = \frac{1}{\eta} \mathbf{k} \times \mathbf{E} \]
\[ \mathbf{E} = -\eta \mathbf{k} \times \mathbf{H} \]

**Maxwell’s Equations for Time-Harmonic Fields**
\[ \nabla \cdot \mathbf{E} = 0 \]
\[ \nabla \times \mathbf{E} = -j\omega \mu \mathbf{H} \]
\[ \nabla \cdot \mathbf{H} = 0 \]
\[ \nabla \times \mathbf{H} = j\omega \varepsilon_c \mathbf{E} \]

**Lossy Medium**
\[ \alpha = \omega \left\{ \frac{\mu \varepsilon'}{2} \left[ \sqrt{1 + \left( \frac{\varepsilon''}{\varepsilon'} \right)^2} - 1 \right] \right\}^{1/2} \quad (\text{Np/m}) \]
\[ \beta = \omega \left\{ \frac{\mu \varepsilon'}{2} \left[ \sqrt{1 + \left( \frac{\varepsilon''}{\varepsilon'} \right)^2} + 1 \right] \right\}^{1/2} \quad (\text{rad/m}) \]
\[ \eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon'}} \left( 1 - j \frac{\varepsilon''}{\varepsilon'} \right)^{-1/2} \quad (\Omega) \]
\[ \delta_c = \frac{1}{\alpha} \quad (\text{m}) \]

**Power Density**
\[ S_{av} = \frac{1}{2} \Re \left[ \mathbf{E} \times \mathbf{H}^* \right] \quad (\text{W/m}^2) \]
Quarter review this Thursday
Final next Friday