Problem 6.6  The square loop shown in Fig. P6.6 is coplanar with a long, straight wire carrying a current

\[ I(t) = 5 \cos(2\pi \times 10^4 t) \quad (A). \]

(a) Determine the emf induced across a small gap created in the loop.
(b) Determine the direction and magnitude of the current that would flow through a 4-\( \Omega \) resistor connected across the gap. The loop has an internal resistance of 1 \( \Omega \).

![Diagram of a square loop coplanar with a long wire](image)

Figure P6.6: Loop coplanar with long wire (Problem 6.6).

Solution:
(a) The magnetic field due to the wire is

\[ \mathbf{B} = \hat{z} \frac{\mu_0 I}{2\pi r} = -\hat{z} \frac{\mu_0 I}{2\pi r}, \]

where in the plane of the loop, \( \hat{z} = -\hat{x} \) and \( r = y \). The flux passing through the loop is

\[
\Phi = \int_{S} \mathbf{B} \cdot d\mathbf{s} = \int_{S} \left( -\hat{z} \frac{\mu_0 I}{2\pi r} \right) \cdot \left[ -\hat{z} 10 \text{ cm} \right] d\mathbf{y}
\]

\[
= \frac{\mu_0 I}{2\pi} \int_{S} \frac{15}{10} d\mathbf{y}
\]

\[
= \frac{4\pi \times 10^{-7} \times 5 \cos(2\pi \times 10^4 t) \times 10^{-1}}{2\pi}
\]

\[
= 1.1 \times 10^{-7} \cos(2\pi \times 10^4 t) \quad \text{(Wb)}.
\]

\[
V_{\text{ind}} = \frac{d\Phi}{dt} = 1.1 \times 2\pi \times 10^4 \sin(2\pi \times 10^4 t) \times 10^{-7}
\]

\[
= 6.9 \times 10^{-3} \sin(2\pi \times 10^4 t) \quad \text{(V)}.
\]

(b) \[
I_{\text{ind}} = \frac{V_{\text{ind}}}{4 + 1} = \frac{6.9 \times 10^{-3}}{5} \sin(2\pi \times 10^4 t) = 1.38 \sin(2\pi \times 10^4 t) \quad \text{(mA)}.
\]

At \( t = 0 \), \( B \) is a maximum, it points in \(-\hat{z}\)-direction, and since it varies as \( \cos(2\pi \times 10^4 t) \), it is decreasing. Hence, the induced current has to be CCW when looking down on the loop, as shown in the figure.
Problem 6.9  A rectangular conducting loop 5 cm × 10 cm with a small air gap in one of its sides is spinning at 7200 revolutions per minute. If the field $B$ is normal to the loop axis and its magnitude is $6 \times 10^{-6}$ T, what is the peak voltage induced across the air gap?

Solution:

\[
\omega = \frac{2\pi \text{ rad/cycle} \times 7200 \text{ cycles/min}}{60 \text{ s/min}} = 240\pi \text{ rad/s},
\]

\[
A = \frac{5 \text{ cm} \times 10 \text{ cm}}{(100 \text{ cm/m})^2} = 5.0 \times 10^{-3} \text{ m}^2.
\]

From Eqs. (6.36) or (6.38), $V_{\text{emf}} = A\omega B_0 \sin \omega t$; it can be seen that the peak voltage is

\[
V_{\text{peak}} = A\omega B_0 = 5.0 \times 10^{-3} \times 240\pi \times 6 \times 10^{-6} = 22.62 \mu\text{V}.
\]
Problem 6.18  An electromagnetic wave propagating in seawater has an electric field with a time variation given by \( \mathbf{E} = 2E_0 \cos \omega t \). If the permittivity of water is \( \varepsilon_0 = 81 \) and its conductivity is \( \sigma = 4 \) (S/m), find the ratio of the magnitudes of the conduction current density to displacement current density at each of the following frequencies:

(a) 1 kHz
(b) 1 MHz
(c) 1 GHz
(d) 100 GHz

**Solution:** From Eq. (6.44), the displacement current density is given by

\[
\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t} = \imath \frac{\partial \mathbf{E}}{\partial t}
\]

and, from Eq. (4.67), the conduction current is \( \mathbf{J} = \sigma \mathbf{E} \). Converting to phasors and taking the ratio of the magnitudes,

\[
\left| \frac{\mathbf{J}}{\mathbf{J}_d} \right| = \left| \frac{\sigma \mathbf{E}}{\omega \varepsilon_0 \mathbf{E}} \right| = \frac{\sigma}{\omega \varepsilon_0}.
\]

(a) At \( f = 1 \text{ kHz} \), \( \omega = 2\pi \times 10^3 \text{ rad/s} \), and

\[
\left| \frac{\mathbf{J}}{\mathbf{J}_d} \right| = \frac{4}{2\pi \times 10^3 \times 81 \times 8.854 \times 10^{-12}} = 888 \times 10^3.
\]

The displacement current is negligible.

(b) At \( f = 1 \text{ MHz} \), \( \omega = 2\pi \times 10^6 \text{ rad/s} \), and

\[
\left| \frac{\mathbf{J}}{\mathbf{J}_d} \right| = \frac{4}{2\pi \times 10^6 \times 81 \times 8.854 \times 10^{-12}} = 888.
\]

The displacement current is practically negligible.

(c) At \( f = 1 \text{ GHz} \), \( \omega = 2\pi \times 10^9 \text{ rad/s} \), and

\[
\left| \frac{\mathbf{J}}{\mathbf{J}_d} \right| = \frac{4}{2\pi \times 10^9 \times 81 \times 8.854 \times 10^{-12}} = 0.888.
\]

Neither the displacement current nor the conduction current are negligible.

(d) At \( f = 100 \text{ GHz} \), \( \omega = 2\pi \times 10^{11} \text{ rad/s} \), and

\[
\left| \frac{\mathbf{J}}{\mathbf{J}_d} \right| = \frac{4}{2\pi \times 10^{11} \times 81 \times 8.854 \times 10^{-12}} = 8.88 \times 10^{-3}.
\]

The conduction current is practically negligible.
**Problem 6.22** If we were to characterize how good a material is as an insulator by its resistance to dissipating charge, which of the following two materials is the better insulator?

- **Dry Soil:** \( \varepsilon_i = 2.5, \quad \sigma = 10^{-4} \text{ (S/m)} \)
- **Fresh Water:** \( \varepsilon_i = 80, \quad \sigma = 10^{-3} \text{ (S/m)} \)

**Solution:** Relaxation time constant \( \tau = \frac{\varepsilon_i}{\sigma} \).

For dry soil, \( \tau = \frac{2.5}{10^{-4}} = 2.5 \times 10^4 \text{ s} \).

For fresh water, \( \tau = \frac{80}{10^{-3}} = 8 \times 10^4 \text{ s} \).

Since it takes longer for charge to dissipate in fresh water, it is a better insulator than dry soil.
Problem 6.27  A Hertzian dipole is a short conducting wire carrying an approximately constant current over its length \( I \). If such a dipole is placed along the \( z \)-axis with its midpoint at the origin, and if the current flowing through it is \( i(t) = I_0 \cos \omega t \), find the following:

(a) The retarded vector potential \( \vec{A}(R, \theta, \phi) \) at an observation point \( Q(R, \theta, \phi) \) in a spherical coordinate system.

(b) The magnetic field phasor \( \vec{H}(R, \theta, \phi) \).

Assume \( l \) to be sufficiently small so that the observation point is approximately equidistant to all points on the dipole; that is, assume \( R' \approx R \).

Solution:

(a) In phasor form, the current is given by \( \vec{I} = I_0 \). Explicitly writing the volume integral in Eq. (6.84) as a double integral over the wire cross section and a single integral over its length,

\[
\vec{A} = \frac{\mu}{4\pi} \int_{-l/2}^{l/2} \int s \, \vec{J}(R_i) e^{-jkR'} \, dr \, dz,
\]

where \( s \) is the wire cross section. The wire is infinitesimally thin, so that \( R' \) is not a function of \( x \) or \( y \) and the integration over the cross section of the wire applies only to the current density. Recognizing that \( \vec{J} = \hat{z}I_0/s \), and employing the relation \( R' \approx R \),

\[
\vec{A} = \hat{z} \frac{\mu I_0}{4\pi} \int_{-l/2}^{l/2} e^{-jkR} \, dz \approx \hat{z} \frac{\mu I_0}{4\pi} \int_{-l/2}^{l/2} e^{-jkR} \, dz = \hat{z} \frac{\mu I_0 l}{4\pi R} e^{-jkR}.
\]

In spherical coordinates, \( \hat{z} = \hat{R} \cos \theta - \hat{\theta} \sin \theta \), and therefore

\[
\vec{A} = (\hat{R} \cos \theta - \hat{\theta} \sin \theta) \frac{\mu I_0 l}{4\pi R} e^{-jkR}.
\]

(b) From Eq. (6.85),

\[
\begin{align*}
\vec{H} &= \frac{1}{\mu} \nabla \times \vec{A} = \frac{k \mu I_0 l}{4\pi} \nabla \times \left[ \left( \hat{R} \cos \theta - \hat{\theta} \sin \theta \right) e^{-jkR} \right] \\
&= \frac{k \mu I_0 l}{4\pi} \left( \hat{R} \frac{\partial}{\partial R} (- \sin \theta e^{-jkR}) - \hat{\theta} \frac{\partial}{\partial \theta} \left( \cos \theta e^{-jkR} \right) \right) \\
&= \frac{k \mu I_0 l \sin \theta e^{-jkR}}{4\pi R} \left( jk + \frac{1}{R} \right).
\end{align*}
\]