EE 135, Winter 2013

Reading: Chapters 1-7

Homework # 7: problems, 7.6, 7.11, 7.26, 7.32
Due March 14, today

FINAL EXAM: March 22, 12-3pm/next Friday
Will cover chapters 1-7

Lecture 18
EE 135/ Winter 2013/

Midterm #2

max = 47

median = 48

\[ z = \frac{x - \mu}{\sigma} \]

standard dev. = 10.2

max = 97

median = 48
EE 135 Cheat Sheet Information

The following tables and charts will be given to you for the final exam:

*Chapter relationships that appear at the end of each chapter for chapters 1-7. Including the tables.*

*The front inside cover of the text.*
*The back inside cover of the text*

*Table 3.1, 3.2*
Chapter 1 Relationships

Electric field due to charge $q$ in free space
\[ E = \hat{R} \frac{q}{4\pi \varepsilon_0 R^2} \]

Magnetic field due to current $I$ in free space
\[ B = \hat{\phi} \frac{\mu_0 I}{2\pi r} \]

Plane wave \[ y(x, t) = A e^{-\alpha x} \cos(\omega t - \beta x + \phi_0) \]
- $\alpha = 0$ in lossless medium
- phase velocity $u_p = f \lambda = \frac{\omega}{\beta}$
- $\omega = 2\pi f$; $\beta = 2\pi / \lambda$
- $\phi_0$ = phase reference

Complex numbers
- Euler’s identity
\[ e^{j\theta} = \cos \theta + j \sin \theta \]
- Rectangular-polar relations
\[ x = |z| \cos \theta, \quad y = |z| \sin \theta, \quad |z| = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}(y/x) \]

Phasor-domain equivalents
Table 1-5
Chapter 3 Relationships

Distance Between Two Points
\[ d = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2} \]
\[ d = [r_2^2 + r_1^2 - 2r_1 r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2]^{1/2} \]
\[ d = \left\{ R_2^2 + R_1^2 - 2R_1 R_2 \left[ \cos \theta_2 \cos \theta_1 + \sin \theta_1 \sin \theta_2 \cos(\phi_2 - \phi_1) \right] \right\}^{1/2} \]

Coordinate Systems Table 3-1
Coordinate Transformations Table 3-2

Vector Products
\[ \mathbf{A} \cdot \mathbf{B} = AB \cos \theta_{AB} \]
\[ \mathbf{A} \times \mathbf{B} = \hat{n} AB \sin \theta_{AB} \]
\[ \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \]
\[ \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \]

Divergence Theorem
\[ \int \nabla \cdot \mathbf{E} \, dV = \oint \mathbf{E} \cdot d\mathbf{s} \]

Vector Operators
\[ \nabla T = \hat{x} \frac{\partial T}{\partial x} + \hat{y} \frac{\partial T}{\partial y} + \hat{z} \frac{\partial T}{\partial z} \]
\[ \nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \]
\[ \nabla \times \mathbf{B} = \hat{x} \left( \frac{\partial B_y}{\partial y} - \frac{\partial B_z}{\partial z} \right) + \hat{y} \left( \frac{\partial B_z}{\partial z} - \frac{\partial B_x}{\partial x} \right) + \hat{z} \left( \frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y} \right) \]
\[ \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \]

(see back cover for cylindrical and spherical coordinates)

Stokes’s Theorem
\[ \int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_C \mathbf{B} \cdot d\mathbf{l} \]
Chapter 4 Relationships
Maxwell’s Equations for Electrostatics

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Electric Field

- **Current density** \( \mathbf{J} = \rho_v \mathbf{u} \)
- **Poisson’s equation** \( \nabla^2 V = -\frac{\rho_v}{\varepsilon} \)
- **Laplace’s equation** \( \nabla^2 V = 0 \)
- **Resistance** \( R = \frac{\int_l \mathbf{E} \cdot d\mathbf{l}}{\oint_s \mathbf{E} \cdot d\mathbf{s}} \)
- **Boundary conditions** Table 4-3
- **Capacitance** \( C = \frac{\oint_s \mathbf{E} \cdot d\mathbf{s}}{-\int_l \mathbf{E} \cdot d\mathbf{l}} \)
- **RC relation** \( RC = \frac{\varepsilon}{\sigma} \)
- **Energy density** \( w_v = \frac{1}{2} \varepsilon \mathbf{E}^2 \)

Point charge
\( \mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi \varepsilon R^2} \)

- **Many point charges** \( \mathbf{E} = \frac{\sum_{i=1}^{N} q_i \left( \mathbf{R} - \mathbf{R}_i \right)}{4\pi \varepsilon \left| \mathbf{R} - \mathbf{R}_i \right|^3} \)

Volume distribution
\( \mathbf{E} = \frac{1}{4\pi \varepsilon} \int_{V'} \frac{\mathbf{R}' \cdot \rho_v \, dV'}{R'^2} \)

Surface distribution
\( \mathbf{E} = \frac{1}{4\pi \varepsilon} \int_{S'} \hat{\mathbf{R}}' \cdot \rho_s \, ds' \)

Line distribution
\( \mathbf{E} = \frac{1}{4\pi \varepsilon} \int_{l'} \hat{\mathbf{R}}' \cdot \rho_v \, dl' \)

Infinite sheet of charge
\( \mathbf{E} = \hat{\mathbf{z}} \frac{\rho_s}{2\varepsilon_0} \)

Infinite line of charge
\( \mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0} = \hat{\mathbf{r}} \frac{D_r}{\varepsilon_0} = \hat{\mathbf{r}} \frac{\rho_v}{2\pi \varepsilon_0 r} \)

Dipole
\( \mathbf{E} = \frac{q d}{4\pi \varepsilon_0 R^3} \left( \mathbf{R} \cos \theta + \hat{\mathbf{\theta}} \sin \theta \right) \)

Relation to \( \nabla V \)
\( \mathbf{E} = -\nabla V \)
Chapter 5 Relationships

Maxwell’s Magnetostatics Equations

Gauss’s Law for Magnetism
\[ \nabla \cdot \mathbf{B} = 0 \quad \iff \quad \oint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{s} = 0 \]

Ampère’s Law
\[ \nabla \times \mathbf{H} = \mathbf{J} \quad \iff \quad \oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{l} = I \]

Lorentz Force on Charge \( q \)
\[ \mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \]

Magnetic Force on Wire
\[ F_m = I \oint_{\mathcal{C}} d\mathbf{l} \times \mathbf{B} \] (N)

Magnetic Torque on Loop
\[ T = m \times \mathbf{B} \] (N·m)
\[ m = \hat{n} N I A \] (A·m²)

Biot–Savart Law
\[ \mathbf{H} = \frac{I}{4\pi} \int_{\mathcal{L}} d\mathbf{l} \times \frac{\hat{R}}{R^2} \] (A/m)

Magnetic Field

Ininitely Long Wire
\[ \mathbf{B} = \hat{\phi} \frac{\mu_0 I}{2\pi r} \] (Wb/m²)

Circular Loop
\[ \mathbf{H} = \hat{\mathbf{z}} \frac{I a^2}{2(a^2 + z^2)^{3/2}} \] (A/m)

Solenoid
\[ \mathbf{B} \approx \hat{\mathbf{z}} \mu_0 n I = \frac{\hat{\mathbf{z}} \mu_0 N I}{l} \] (Wb/m²)

Vector Magnetic Potential
\[ \mathbf{B} = \nabla \times \mathbf{A} \] (Wb/m²)

Vector Poisson’s Equation
\[ \nabla^2 \mathbf{A} = -\mu \mathbf{J} \]

Inductance
\[ L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{1}{l} \int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{s} \] (H)

Magnetic Energy Density
\[ w_m = \frac{1}{2} \mu H^2 \] (J/m³)
Summary

Lenz’s Law: Induced current in a loop is always in a direction that produces a magnetic field in a direction that opposes the change in magnetic flux.
Summary

Chapter 7 Relationships

### Complex Permittivity
\[
\varepsilon_c = \varepsilon' - j \varepsilon'' \\
\varepsilon' = \varepsilon \\
\varepsilon'' = \frac{\sigma}{\omega}
\]

### Lossless Medium
\[
k = \omega \sqrt{\mu \varepsilon} \\
\eta = \sqrt{\frac{\mu}{\varepsilon}} \quad (\Omega) \\
u_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu \varepsilon}} \quad (\text{m/s}) \\
\lambda = \frac{2\pi}{k} = \frac{u_p}{f} \quad (\text{m})
\]

### Wave Polarization
\[
\vec{H} = \frac{1}{\eta} \hat{k} \times \vec{E} \\
\vec{E} = -\eta \hat{k} \times \vec{H}
\]

### Maxwell’s Equations for Time-Harmonic Fields
\[
\nabla \cdot \vec{E} = 0 \\
\nabla \times \vec{E} = -j \omega \mu \vec{H} \\
\nabla \cdot \vec{H} = 0 \\
\nabla \times \vec{H} = j \omega \varepsilon_c \vec{E}
\]

### Lossy Medium
\[
\alpha = \omega \left\{ \frac{\mu \varepsilon'}{2} \left[ \sqrt{1 + \left( \frac{\varepsilon''}{\varepsilon'} \right)^2} - 1 \right] \right\}^{1/2} \quad (\text{Np/m}) \\
\beta = \omega \left\{ \frac{\mu \varepsilon'}{2} \left[ \sqrt{1 + \left( \frac{\varepsilon''}{\varepsilon'} \right)^2} + 1 \right] \right\}^{1/2} \quad (\text{rad/m}) \\
\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon'}} \left( 1 - \frac{j \varepsilon''}{\varepsilon'} \right)^{-1/2} \quad (\Omega) \\
\delta_e = \frac{1}{\alpha} \quad (\text{m})
\]

### Power Density
\[
S_{\text{av}} = \frac{1}{2} \Re \left[ \vec{E} \times \vec{H}^* \right] \quad (\text{W/m}^2)
\]
## Chapter 2 Relationships

### TEM Transmission Lines

\[ L'C' = \mu \varepsilon \]
\[ G' \]
\[ C' = \frac{\sigma}{\varepsilon} \]

\[ \alpha = \Re(\gamma) = \Re \left( \sqrt{(R' + j\omega L')(G' + j\omega C')} \right) \quad \text{(Np/m)} \]
\[ \beta = \Im(\gamma) = \Im \left( \sqrt{(R' + j\omega L')(G' + j\omega C')} \right) \quad \text{(rad/m)} \]

\[ Z_0 = \frac{R' + j\omega L'}{\gamma} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \quad \Omega \]

\[ \Gamma = \frac{z_L - 1}{z_L + 1} \]

### Step Function Transient Response

\[ V_i^+ = \frac{V_0 Z_0}{R_g + Z_0} \]
\[ V_\infty = \frac{V_0 R_L}{R_g + R_L} \]
\[ \Gamma_g = \frac{R_g - Z_0}{R_g + Z_0} \]
\[ \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} \]

### Lossless Line

\[ \alpha = 0 \]
\[ \beta = \omega \sqrt{L'C'} \]

\[ Z_0 = \sqrt{\frac{L'}{C'}} \]

\[ u_p = \frac{1}{\sqrt{\mu \varepsilon}} \quad \text{(m/s)} \]

\[ \lambda = \frac{u_p}{f} = \frac{c}{f} \cdot \frac{1}{\sqrt{\varepsilon_r}} = \frac{\lambda_0}{\sqrt{\varepsilon_r}} \]

\[ d_{\min} = \frac{\theta_r \lambda}{4\pi} + \frac{(2n + 1)\lambda}{4} \]
\[ S = \frac{1 + |\Gamma|}{1 - |\Gamma|} \]
\[ P_{av} = \frac{|V_0^+|^2}{2Z_0} \left[ 1 - |\Gamma|^2 \right] \]
Standing Waves

Using the relation $V_0^- = \Gamma V_0^+$ in Eqs. (2.5) yields

$$\tilde{V}(z) = V_0^+ (e^{-j\beta z} + \Gamma e^{j\beta z}),$$

$$\tilde{I}(z) = \frac{V_0^+}{Z_0} (e^{-j\beta z} - \Gamma e^{j\beta z}).$$
trans lines can look like diff. reactive elements depending on the char. and lengths.

we looked at some special cases: (harmonic wave as t → 0)

1. short circuit line. $Z_L = 0, \Gamma = -1$

   \[ Z_{in} = \begin{cases} \frac{Z_0 \tan \beta Z}{j} & \text{purely reactive} \\ -\frac{Z_0 \tan \beta Z}{j} & \text{inductive} \end{cases} \]

   \[ \frac{Z_{sc}(z)}{Z_0} \]

   \[ \beta = \frac{2\pi}{\lambda} \]

   \[ Z_{in} = \begin{cases} jZ_0 \tan \beta Z & \text{purely reactive} \\ -jZ_0 \tan \beta Z & \text{inductive} \end{cases} \]

2. open circuit line $Z_L = \infty, \Gamma = +1$

   \[ \frac{Z_{oc}(z)}{Z_0} \]

   \[ \beta = \frac{2\pi}{\lambda} \]

   \[ Z_{in} = \begin{cases} jZ_0 \tan \beta Z & \text{purely reactive} \\ -jZ_0 \tan \beta Z & \text{inductive} \end{cases} \]

   \[ \text{and we could use this info to} \]

   \[ \begin{array}{c}
   \text{analyze the reactivity of the trans line.}
   \end{array} \]
and we could use this info to 
measure the properties of the trans line 

take 2o and phase unit. \( \beta = \alpha \sqrt{L/C} \) for broken line \( \sqrt{\frac{L}{C}} \) for lossless line.

so thus \( Z_0(\beta) = \sqrt{Z_{in} \cdot Z_{oc}} \)

\( \tan \beta = \sqrt{-\frac{Z_{oc}}{Z_{in}}} \)
Chapter 3 Relationships

Distance Between Two Points
\[ d = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2} \]
\[ d = [r_2^2 + r_1^2 - 2r_1 r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2]^{1/2} \]
\[ d = \left\{ R_2^2 + R_1^2 - 2 R_1 R_2 \cos(\theta_2 \cos \theta_1 + \sin \theta_1 \sin \theta_2 \cos(\phi_2 - \phi_1)) \right\}^{1/2} \]

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Divergence Theorem
\[ \int_{V} \nabla \cdot \mathbf{E} \, dV = \oint_{S} \mathbf{E} \cdot d\mathbf{s} \]

Vector Operators
\[ \nabla T = \hat{x} \frac{\partial T}{\partial x} + \hat{y} \frac{\partial T}{\partial y} + \hat{z} \frac{\partial T}{\partial z} \]
\[ \nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \]
\[ \nabla \times \mathbf{B} = \hat{x} \left( \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) + \hat{y} \left( \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) + \hat{z} \left( \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \]
\[ \nabla^2 \mathbf{V} = \frac{\partial^2 \mathbf{V}}{\partial x^2} + \frac{\partial^2 \mathbf{V}}{\partial y^2} + \frac{\partial^2 \mathbf{V}}{\partial z^2} \]
(see back cover for cylindrical and spherical coordinates)

Stokes’s Theorem
\[ \int_{S} (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_{C} \mathbf{B} \cdot d\mathbf{l} \]
Chapter 4 Relationships

Maxwell’s Equations for Electrostatics

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Electric Field

- **Current density**
  \[ \mathbf{J} = \rho_v \mathbf{u} \]

- **Poisson’s equation**
  \[ \nabla^2 V = -\frac{\rho_v}{\varepsilon} \]

- **Laplace’s equation**
  \[ \nabla^2 V = 0 \]

- **Resistance**
  \[ R = \frac{\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_s \varepsilon \mathbf{E} \cdot d\mathbf{s}} \]

- **Boundary conditions**
  Table 4-3

- **Capacitance**
  \[ C = \frac{\int_s \varepsilon \mathbf{E} \cdot d\mathbf{s}}{\int_l \mathbf{E} \cdot d\mathbf{l}} \]

- **RC relation**
  \[ RC = \frac{\varepsilon}{\sigma} \]

- **Energy density**
  \[ w_e = \frac{1}{2} \varepsilon E^2 \]

Point charge
\[ \mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi \varepsilon R^2} \]

- **Many point charges**
  \[ \mathbf{E} = \frac{1}{4\pi \varepsilon} \sum_{i=1}^{N} \frac{q_i (\mathbf{R} - \mathbf{R}_i)}{|\mathbf{R} - \mathbf{R}_i|^3} \]

Volume distribution
\[ \mathbf{E} = \frac{1}{4\pi \varepsilon} \int_{V'} \hat{\mathbf{R}}' \frac{\rho_v dV'}{R'^2} \]

Surface distribution
\[ \mathbf{E} = \frac{1}{4\pi \varepsilon} \int_{S'} \hat{\mathbf{R}}' \frac{\rho_s ds'}{R'^2} \]

Line distribution
\[ \mathbf{E} = \frac{1}{4\pi \varepsilon} \int_{L'} \hat{\mathbf{R}}' \frac{\rho_l dl'}{R'^2} \]

Infinite sheet of charge
\[ \mathbf{E} = \hat{\mathbf{z}} \frac{\rho_s}{2\varepsilon_0} \]

Infinite line of charge
\[ \mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0} = \hat{\mathbf{r}} \frac{D_r}{\varepsilon_0} = \hat{\mathbf{r}} \frac{\rho_l}{2\pi \varepsilon_0 r} \]

Dipole
\[ \mathbf{E} = \frac{q d}{4\pi \varepsilon_0 R^3} (\hat{\mathbf{R}} \cos \theta + \hat{\mathbf{\theta}} \sin \theta) \]

Relation to \( V \)
\[ \mathbf{E} = -\nabla V \]
Maxwell’s Equations

\[
\begin{align*}
\nabla \cdot \mathbf{D} &= \rho_v, \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\
\nabla \cdot \mathbf{B} &= 0, \\
\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.
\end{align*}
\]

Under \textit{static} conditions, none of the quantities appearing in Maxwell’s equations are functions of time (i.e., \(\partial / \partial t = 0\)). \textit{This happens when all charges are permanently fixed in space, or, if they move, they do so at a steady rate so that} \(\rho_v\) \textit{and} \(\mathbf{J}\) \textit{are constant in time}. Under these circumstances, the time derivatives of \(\mathbf{B}\) and \(\mathbf{D}\) in Eqs. (4.1b) and (4.1d) vanish, and Maxwell’s equations reduce to

\textit{Electrostatics}

\[
\begin{align*}
\nabla \cdot \mathbf{D} &= \rho_v, \quad (4.2a) \\
\nabla \times \mathbf{E} &= 0. \quad (4.2b)
\end{align*}
\]

\textit{Magnetostatics}

\[
\begin{align*}
\nabla \cdot \mathbf{B} &= 0, \quad (4.3a) \\
\nabla \times \mathbf{H} &= \mathbf{J}. \quad (4.3b)
\end{align*}
\]

\textit{Electric and magnetic fields become decoupled under static conditions.}
\[ \nabla \cdot \mathbf{D} = \rho_v \]

MAXWELL'S FIRST EQUATION IS GAUSS'S LAW. IT SAYS THAT ELECTRIC FIELD LINES DIVERGE FROM POSITIVE CHARGES AND CONVERGE TO NEGATIVE CHARGES.

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]

THE SECOND EQUATION IS FARADAY'S LAW: ELECTRIC FIELD LINES CURL AROUND CHANGING MAGNETIC FIELDS. CHANGING MAGNETIC FIELDS INDUCE ELECTRIC FIELDS.

From Chalmers.E&M
\[ \nabla \cdot \mathbf{B} = 0 \]

The third equation says that magnetic fields never diverge or converge. They always go in closed curves.

\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \]

Finally, the last equation says that magnetic field lines curl around electric currents. We have seen that a magnetic field circles around a conducting wire.

...and here Maxwell had a critical brainstorm! (An electrical storm, of course!)
Charge Distributions

Volume charge density:
\[ \rho_v = \lim_{\Delta V \to 0} \frac{\Delta q}{\Delta V} = \frac{dq}{dV} \quad \text{(C/m}^3\text{)} \]

Total Charge in a Volume
\[ Q = \int_V \rho_v \, dV \quad \text{(C)} \]

Surface and Line Charge Densities
\[ \rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds} \quad \text{(C/m}^2\text{)} \]
\[ \rho_\ell = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl} \quad \text{(C/m)} \]
Current Density

The amount of charge that crosses the tube’s cross-sectional surface $\Delta s'$ in time $\Delta t$ is therefore

$$\Delta q' = \rho_v \Delta V = \rho_v \Delta l \Delta s' = \rho_v u \Delta s' \Delta t. \quad (4.8)$$

For a surface with any orientation:

$$\Delta q = \rho_v u \cdot \Delta s \Delta t, \quad (4.9)$$

where $\Delta s = \hat{n} \Delta s$ and the corresponding total current flowing in the tube is

$$\Delta I = \frac{\Delta q}{\Delta t} = \rho_v u \cdot \Delta s = J \cdot \Delta s, \quad (4.10)$$

where

$$J = \rho_v u \quad (A/m^2) \quad (4.11)$$

$J$ is called the current density.

**Figure 4-2:** Charges with velocity $u$ moving through a cross section $\Delta s'$ in (a) and $\Delta s$ in (b).

When a current is due to the actual movement of electrically charged matter, it is called a convection current, and $J$ is called a convection current density.
Convection vs. Conduction

When a current is due to the movement of charged particles relative to their host material, \( J \) is called a \textit{conduction current density}.

This movement of electrons from atom to atom constitutes a \textit{conduction current}. The electrons that emerge from the wire are not necessarily the same electrons that entered the wire at the other end.

Conduction current, which is discussed in more detail in Section 4-6, obeys Ohm’s law, whereas convection current does not.
Aurora Borealis
Coulomb’s Law

Electric field at point $P$ due to single charge

$$ E = \hat{R} \frac{q}{4\pi \varepsilon R^2} \quad \text{(V/m)} $$

Electric force on a test charge placed at $P$

$$ F = q' E \quad \text{(N)} $$

Electric flux density $D$

$$ D = \varepsilon E $$

$$ \varepsilon = \varepsilon_r \varepsilon_0, $$

If $\varepsilon$ is independent of the magnitude of $E$, then the material is said to be linear because $D$ and $E$ are related linearly, and if it is independent of the direction of $E$, the material is said to be isotropic.

$$ \varepsilon_0 = 8.85 \times 10^{-12} \simeq (1/36\pi) \times 10^{-9} \quad \text{(F/m)} $$
Electric Field Due to 2 Charges

with $R$, the distance between $q_1$ and $P$, replaced with $|\mathbf{R} - \mathbf{R}_1|$ and the unit vector $\hat{\mathbf{R}}$ replaced with $(\mathbf{R} - \mathbf{R}_1)/|\mathbf{R} - \mathbf{R}_1|$. Thus,

$$E_1 = \frac{q_1(\mathbf{R} - \mathbf{R}_1)}{4\pi \varepsilon |\mathbf{R} - \mathbf{R}_1|^3} \quad (\text{V/m}).$$  

(4.17a)

Similarly, the electric field at $P$ due to $q_2$ alone is

$$E_2 = \frac{q_2(\mathbf{R} - \mathbf{R}_2)}{4\pi \varepsilon |\mathbf{R} - \mathbf{R}_2|^3} \quad (\text{V/m}).$$  

(4.17b)

The electric field obeys the principle of linear superposition.

Hence, the total electric field $\mathbf{E}$ at $P$ due to $q_1$ and $q_2$ is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$$

$$= \frac{1}{4\pi \varepsilon} \left[ \frac{q_1(\mathbf{R} - \mathbf{R}_1)}{|\mathbf{R} - \mathbf{R}_1|^3} + \frac{q_2(\mathbf{R} - \mathbf{R}_2)}{|\mathbf{R} - \mathbf{R}_2|^3} \right].$$  

(4.18)

Figure 4-4: The electric field $\mathbf{E}$ at $P$ due to two charges is equal to the vector sum of $\mathbf{E}_1$ and $\mathbf{E}_2$. 
Electric Field due to Multiple Charges

**Example 4-3: Electric Field Due to Two Point Charges**

Two point charges with \( q_1 = 2 \times 10^{-5} \) C and \( q_2 = -4 \times 10^{-5} \) C are located in free space at points with Cartesian coordinates \((1, 3, -1)\) and \((-3, 1, -2)\), respectively. Find (a) the electric field \( E \) at \((3, 1, -2)\) and (b) the force on a \( 8 \times 10^{-5} \) C charge located at that point. All distances are in meters.

**Solution:** (a) From Eq. (4.18), the electric field \( E \) with \( \varepsilon = \varepsilon_0 \) (free space) is

\[
E = \frac{1}{4\pi \varepsilon_0} \left[ q_1 \frac{\mathbf{R} - \mathbf{R}_1}{|\mathbf{R} - \mathbf{R}_1|^3} + q_2 \frac{\mathbf{R} - \mathbf{R}_2}{|\mathbf{R} - \mathbf{R}_2|^3} \right] \quad \text{(V/m)}.
\]

The vectors \( \mathbf{R}_1, \mathbf{R}_2, \) and \( \mathbf{R} \) are

\[
\mathbf{R}_1 = \hat{x} + \hat{y}3 - \hat{z}, \\
\mathbf{R}_2 = -\hat{x}3 + \hat{y} - \hat{z}2, \\
\mathbf{R} = \hat{x}3 + \hat{y} - \hat{z}2.
\]

Hence,

\[
E = \frac{1}{4\pi \varepsilon_0} \left[ \frac{2(\hat{x}2 - \hat{y}2 - \hat{z})}{27} - \frac{4(\hat{x}6)}{216} \right] \times 10^{-5}
\]

\[
= \frac{\hat{x} - \hat{y}4 - \hat{z}2}{108\pi \varepsilon_0} \times 10^{-5} \quad \text{(V/m)}.
\]

(b)

\[
F = q_3 E = 8 \times 10^{-5} \times \frac{\hat{x} - \hat{y}4 - \hat{z}2}{108\pi \varepsilon_0} \times 10^{-5}
\]

\[
= \frac{\hat{x}2 - \hat{y}8 - \hat{z}4}{27\pi \varepsilon_0} \times 10^{-10} \quad \text{(N)}.
\]
Electric Field Due to Charge Distributions

Field due to:

A differential amount of charge \( dq = \rho_v \, dV' \) contained in a
differential volume \( dV' \) is

\[
d\mathbf{E} = \hat{\mathbf{R}}' \frac{dq}{4\pi \varepsilon R'^2} = \hat{\mathbf{R}}' \frac{\rho_v \, dV'}{4\pi \varepsilon R'^2}, \quad (4.20)
\]

\[
\mathbf{E} = \int_{V'} d\mathbf{E} = \frac{1}{4\pi \varepsilon} \int_{V'} \hat{\mathbf{R}}' \frac{\rho_v \, dV'}{R'^2}
\]

(volume distribution). \quad (4.21a)

\[ E = \frac{1}{4\pi \varepsilon} \int_{S'} \hat{\mathbf{R}}' \frac{\rho_s \, ds'}{R'^2} \quad \text{(surface distribution)}, \quad (4.21b) \]

\[ E = \frac{1}{4\pi \varepsilon} \int_{L'} \hat{\mathbf{R}}' \frac{\rho_l \, dl'}{R'^2} \quad \text{(line distribution)}. \quad (4.21c) \]
Example 4-4: Electric Field of a Ring of Charge

A ring of charge of radius $b$ is characterized by a uniform line charge density of positive polarity $\rho_\ell$. The ring resides in free space and is positioned in the $x$-$y$ plane as shown in Fig. 4-6. Determine the electric field intensity $\mathbf{E}$ at a point $P = (0, 0, h)$ along the axis of the ring at a distance $h$ from its center.

Solution: We start by considering the electric field generated by a differential ring segment with cylindrical coordinates $(b, \phi, 0)$ in Fig. 4-6(a). The segment has length $dl = b\, d\phi$ and contains charge $dq = \rho_\ell\, dl = \rho_\ell b\, d\phi$. The distance vector $\mathbf{R}_1'$ from segment 1 to point $P = (0, 0, h)$ is

$$\mathbf{R}_1' = -\hat{\mathbf{r}} b + \hat{\mathbf{z}} h,$$

from which it follows that

$$R_1' = |\mathbf{R}_1'| = \sqrt{b^2 + h^2}, \quad \hat{\mathbf{R}}_1 = \frac{\mathbf{R}_1'}{|\mathbf{R}_1'|} = \frac{-\hat{\mathbf{r}} b + \hat{\mathbf{z}} h}{\sqrt{b^2 + h^2}}.$$

The electric field at $P = (0, 0, h)$ due to the charge in segment 1 therefore is

$$d\mathbf{E}_1 = \frac{1}{4\pi \varepsilon_0} \frac{\rho_\ell}{R_1'^2} \frac{dl}{4\pi \varepsilon_0} = \frac{\rho_\ell b}{(b^2 + h^2)^{3/2}}\, d\phi.$$

Figure 4-6: Ring of charge with line density $\rho_\ell$. (a) The field $d\mathbf{E}_1$ due to infinitesimal segment 1 and (b) the fields $d\mathbf{E}_1$ and $d\mathbf{E}_2$ due to segments at diametrically opposite locations (Example 4-4).
\[ d\mathbf{E}_1 = \frac{1}{4\pi \varepsilon_0} \frac{\mathbf{R}'}{R'^2} \mathbf{R}' \frac{\rho \mathbf{e} \cdot dl}{b^2 + h^2} = \frac{\rho \mathbf{e}}{4\pi \varepsilon_0} \frac{(-\mathbf{b} + \hat{z}h)}{(b^2 + h^2)^{3/2}} d\phi. \]

The field \( d\mathbf{E}_1 \) has component \( d\mathbf{E}_{1r} \) along \( -\hat{r} \) and component \( d\mathbf{E}_{1z} \) along \( \hat{z} \). From symmetry considerations, the field \( d\mathbf{E}_2 \) generated by differential segment 2 in Fig. 4-6(b), which is located diametrically opposite to segment 1, is identical to \( d\mathbf{E}_1 \) except that the \( \hat{r} \)-component of \( d\mathbf{E}_2 \) is opposite that of \( d\mathbf{E}_1 \). Hence, the \( \hat{r} \)-components in the sum cancel and the \( \hat{z} \)-contributions add. The sum of the two contributions is

\[ d\mathbf{E} = d\mathbf{E}_1 + d\mathbf{E}_2 = \hat{z} \frac{\rho \mathbf{e} b h}{2\pi \varepsilon_0 (b^2 + h^2)^{3/2}} \frac{d\phi}{b^2 + h^2}. \quad (4.22) \]

Since for every ring segment in the semicircle defined over the azimuthal range \( 0 \leq \phi \leq \pi \) (the right-hand half of the circular ring) there is a corresponding segment located diametrically opposite at \((\phi + \pi)\), we can obtain the total field generated by the ring by integrating Eq. (4.22) over a semicircle as

\[ \mathbf{E} = \hat{z} \frac{\rho \mathbf{e} b h}{2\pi \varepsilon_0 (b^2 + h^2)^{3/2}} \int_0^\pi d\phi \]

\[ = \hat{z} \frac{\rho \mathbf{e} b h}{2\varepsilon_0 (b^2 + h^2)^{3/2}} \]

\[ = \hat{z} \frac{h}{4\pi \varepsilon_0 (b^2 + h^2)^{3/2}} Q, \quad (4.23) \]

where \( Q = 2\pi b \rho \) is the total charge on the ring.
Electric Ring of Charge

\[ \vec{E} = \frac{z \cdot Q}{4\pi \varepsilon_0 (b^2 + z^2)^{3/2}} \]  
\( z = \text{distance from center of ring.} \)

\[ (b^2 + z^2)^{3/2} = z^3 \left(1 + \frac{b^2}{z^2}\right)^{3/2} \]

\[ \text{Using Taylor Series Expansion} \]
\[ f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \cdots \]

\[ \text{Then} (b^2 + z^2)^{-3/2} = z^{-3} \left(1 + \frac{b^2}{z^2}\right)^{-3/2} \]

\[ = z^{-3} \left[1 - \frac{3}{2} \frac{b^2}{z^2} + \cdots \right] \]

\[ = z^{-3} + \mathcal{O}(z^{-5}) \]

\[ \therefore \vec{E} \approx \frac{z \cdot Q}{4\pi \varepsilon_0 z^3} \]

\[ \text{falls off far away like an electric dipole} \]
Gauss’s Law

\[ \nabla \cdot \mathbf{D} = \rho_v \]

(Differential form of Gauss’s law),

\[ \int \nabla \cdot \mathbf{D} \, dV = \int \rho_v \, dV = Q \]

Application of the divergence theorem gives:

\[ \int_{V} \nabla \cdot \mathbf{D} \, dV = \oint_{S} \mathbf{D} \cdot ds. \]  \hspace{1cm} (4.28)

Comparison of Eq. (4.27) with Eq. (4.28) leads to

\[ \oint_{S} \mathbf{D} \cdot ds = Q \]  \hspace{1cm} (4.29)

(Integral form of Gauss’s law).

The integral form of Gauss’s law is illustrated diagrammatically in Fig. 4-8; for each differential surface element \( ds \), \( \mathbf{D} \cdot ds \) is the electric field flux flowing outward of \( V \) through \( ds \), and the total flux through surface \( S \) equals the enclosed charge \( Q \). The surface \( S \) is called a Gaussian surface.

Figure 4-8: The integral form of Gauss’s law states that the outward flux of \( \mathbf{D} \) through a surface is proportional to the enclosed charge \( Q \).
Applying Gauss’s Law

\[ \oint_{s} \mathbf{D} \cdot ds = Q \]  \hspace{1cm} (4.29)

(Integral form of Gauss’s law).

Gauss’s law, as given by Eq. (4.29), provides a convenient method for determining the flux density \( \mathbf{D} \) when the charge distribution possesses symmetry properties that allow us to infer the variations of the magnitude and direction of \( \mathbf{D} \) as a function of spatial location, thereby facilitating the integration of \( \mathbf{D} \) over a cleverly chosen Gaussian surface.

Example 4-6: Electric Field of an Infinite Line Charge

Use Gauss’s law to obtain an expression for \( \mathbf{E} \) due to an infinitely long line with uniform charge density \( \rho_{\ell} \) that resides along the \( z \)-axis in free space.

Construct an imaginary Gaussian cylinder of radius \( r \) and height \( h \):

\[ \int_{z=0}^{h} \int_{\phi=0}^{2\pi} \hat{r} D_{r} \cdot \hat{r} r \ d\phi \ dz = \rho_{\ell} h \]

or

\[ 2\pi h D_{r} r = \rho_{\ell} h, \]

which yields

\[ \mathbf{E} = \frac{\mathbf{D}}{\varepsilon_{0}} = \hat{r} \frac{D_{r}}{\varepsilon_{0}} = \hat{r} \frac{\rho_{\ell}}{2\pi \varepsilon_{0} r} \]  \hspace{1cm} (4.33)

(infinite line charge).
Applying Gauss' Law

spherical shell has charge density:
\[ p_v = -\frac{p_v_0}{R^2} \quad a \leq R \leq b \]

center is charge free, \( p_v > 0 \)
what is \( \vec{D} \) everywhere?

because of sym., we know \( \vec{D} \) is radial
ie, \( \vec{D} = \vec{R} \, \vec{D}_R \) in sph. coord. system.
for any \( R \), \( \int \vec{D} \cdot d\vec{S} = Q \) — enclosed charge

\[ \therefore \int_{S_0} \vec{R} \cdot d\vec{S} = Q \]

\[ \oint_{S_R} \vec{R} \cdot d\vec{S} = Q = D_R \frac{Q}{4\pi R^2} \]

\[ D_R = \frac{Q}{4\pi R^2} \quad \text{outside} \ R \]
Applying Gauss' Law

\[ D_r = \frac{Q}{4\pi R^2} \]

1. \( R < a \), \( Q = 0 \) \( \Rightarrow \) \[ D_r = 0 \]

2. \( R \geq b \) \( Q = \int_{R=a}^{R=b} dV \rho \)

In spherical coordinates, \( dV = R^2 \sin \theta \, dR \, d\theta \, d\phi \)

\[ Q = \int_{R=a}^{R=b} \left[ -\frac{\rho_0}{R^2} \right] R^2 \sin \theta \, dR \, d\theta \, d\phi \]

\[ = 2\pi \rho_0 \int_a^b dR \int_0^{2\pi} d\phi \int_0^\pi \sin \theta \, d\theta \]

\[ = 4\pi \rho_0 (b-a) \]

\[ D_r = -\frac{\rho_0 (b-a)}{R^2} \]

If \( a \leq R \leq b \) then

\[ Q = \int_{R=a}^{R=b} \rho \, dV = \int_{R=a}^{R=b} \left[ -\frac{\rho_0}{R^2} \right] 4\pi R^2 \, dR \]

\[ Q = -\frac{\rho_0}{R^2} (b-a) \]
Electric Scalar Potential

**Figure 4-12:** In electrostatics, the potential difference between $P_2$ and $P_1$ is the same irrespective of the path used for calculating the line integral of the electric field between them.

\[
\oint_{P_1} dV = -\oint_{P_1} E \cdot dl.
\]

\[
V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} E \cdot dl, \quad (4.39)
\]

\[
\oint_{C} E \cdot dl = 0 \quad \text{(Electrostatics)}. \quad (4.40)
\]

A vector field whose line integral along any closed path is zero is called a **conservative** or an **irrotational** field. Hence, the electrostatic field $\mathbf{E}$ is conservative.
Electric Potential Due to Charges

In electric circuits, we usually select a convenient node that we call ground and assign it zero reference voltage. In free space and material media, we choose infinity as reference with $V = 0$. Hence, at a point $P$

$$V_{21} = V_2 - V_1 = - \int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}, \quad (4.39)$$

For a point charge, $V$ at range $R$ is:

$$V = -\int_{\infty}^{R} \left( \hat{\mathbf{R}} \cdot \frac{q}{4\pi \varepsilon R^2} \right) \cdot \hat{\mathbf{R}} \, dR = \frac{q}{4\pi \varepsilon R} \quad (V). \quad (4.45)$$

For continuous charge distributions:

$$V = \frac{1}{4\pi \varepsilon} \int_{V'} \frac{\rho_v}{R} \, dV' \quad \text{(volume distribution),} \quad (4.48a)$$

$$V = \frac{1}{4\pi \varepsilon} \int_{S'} \frac{\rho_s}{R} \, ds' \quad \text{(surface distribution),} \quad (4.48b)$$

$$V = \frac{1}{4\pi \varepsilon} \int_{l'} \frac{\rho_l}{R} \, dl' \quad \text{(line distribution).} \quad (4.48c)$$
Relating $\mathbf{E}$ to $V$

$$dV = -\mathbf{E} \cdot d\mathbf{l}. \quad (4.49)$$

For a scalar function $V$, Eq. (3.73) gives

$$dV = \nabla V \cdot d\mathbf{l}, \quad (4.50)$$

where $\nabla V$ is the gradient of $V$. Comparison of Eq. (4.49) with Eq. (4.50) leads to

$$\mathbf{E} = -\nabla V. \quad (4.51)$$

This differential relationship between $V$ and $\mathbf{E}$ allows us to determine $\mathbf{E}$ for any charge distribution by first calculating $V$ and then taking the negative gradient of $V$ to find $\mathbf{E}$. 
Example 4-7: Electric Field of an Electric Dipole

Solution: To simplify the derivation, we align the dipole along the z-axis and center it at the origin [Fig. 4-13(a)]. For the two charges shown in Fig. 4-13(a), application of Eq. (4.47) gives

\[ V = \frac{1}{4\pi \varepsilon_0} \left( \frac{q}{R_1} + \frac{-q}{R_2} \right) = \frac{q}{4\pi \varepsilon_0} \left( \frac{R_2 - R_1}{R_1 R_2} \right). \]

Since \( d \ll R \), the lines labeled \( R_1 \) and \( R_2 \) in Fig. 4-13(a) are approximately parallel to each other, in which case the following approximations apply:

\[ R_2 - R_1 \simeq d \cos \theta, \quad R_1 R_2 \simeq R^2. \]

Hence,

\[ V = \frac{q d \cos \theta}{4\pi \varepsilon_0 R^2}. \]  \( \text{(4.52)} \)
Example 4-7: Electric Field of an Electric Dipole (cont.)

\[ qd \cos \theta = qd \cdot \mathbf{\hat{R}} = p \cdot \mathbf{\hat{R}}, \]

where \( p = qd \) is called the dipole moment. Using Eq. (4.53) in Eq. (4.52) then gives

\[ V = \frac{p \cdot \mathbf{\hat{R}}}{4\pi \varepsilon_0 R^2} \] \hspace{1cm} \text{(electric dipole).} \hspace{1cm} (4.54)

In spherical coordinates, Eq. (4.51) is given by

\[ \mathbf{E} = -\nabla V \]

\[ = - \left( \mathbf{\hat{R}} \frac{\partial V}{\partial R} + \hat{\theta} \frac{1}{R} \frac{\partial V}{\partial \theta} + \hat{\phi} \frac{1}{R \sin \theta} \frac{\partial V}{\partial \phi} \right), \] \hspace{1cm} (4.55)

\[ \mathbf{E} = \frac{qd}{4\pi \varepsilon_0 R^3} (\mathbf{\hat{R}} 2 \cos \theta + \hat{\theta} \sin \theta) \] \hspace{1cm} \text{(V/m).}
$E$ field of a "dipole". 1

**General case:**

$$E = \frac{Q}{4\pi \varepsilon_0} \left[ \frac{\hat{R}_1}{|\hat{R}_1|^3} - \frac{\hat{R}_2}{|\hat{R}_2|^3} \right]$$

Consider $E(y)$ where $y \gg d$.

have $R_1 = R_2 = \sqrt{d^2 + y^2}$

and because of symmetry, the $\hat{z}$ component is the only one that does not cancel.

$$E_x = E \sin \theta = E \frac{d}{\sqrt{y^2 + d^2}}$$

$$E = \frac{Q}{4\pi \varepsilon_0} \left( \frac{1}{(y^2 + d^2)\sqrt{y^2 + d^2}} \right) \Rightarrow E_x = \frac{Q}{4\pi \varepsilon_0} \frac{1}{(y^2 + d^2)^{3/2}}$$

$$E = -\frac{Q \hat{z}}{2\pi \varepsilon_0} \frac{d}{(y^2 + d^2)^{3/2}} = \frac{-Qd \hat{z}}{2\pi \varepsilon_0 y^3 (1 + \frac{d^2}{y^2})^{3/2}} = -E^\infty$$

Near when $d/y \ll 1$, then

$$\left(1 + \frac{d^2}{y^2}\right)^{-3/2} \approx 1 - \frac{3}{2} \frac{d^2}{y^2} + \ldots$$

$$\therefore E \rightarrow -\frac{Q \hat{z}}{2\pi \varepsilon_0 y^3} + \text{term } O\left(\frac{d^3}{y^5}\right) \ldots \text{ it falls off as } \frac{1}{y^3} \rightarrow$$

**Note:** in text dipole "length" = $d$  

= not $2d$ as here.
E field of Dipole (unit)

\[ \mathbf{E}^0 = \frac{Q}{4\pi \varepsilon \varepsilon_0} \left[ \frac{\hat{\mathbf{z}}}{(z-d)^2} - \frac{\hat{\mathbf{z}}}{(z+d)^2} \right] \]

\[ = \frac{Q \hat{\mathbf{z}}}{4\pi \varepsilon \varepsilon_0} \left[ (z-d)^{-2} - (z+d)^{-2} \right] \]

\[ = \frac{Q \hat{\mathbf{z}}}{4\pi \varepsilon \varepsilon_0} \left[ \left(1 - \frac{d}{z}\right)^{-2} - \left(1 + \frac{d}{z}\right)^{-2} \right] \]

\[ \text{while for } z \to \infty \]

\[ (1 - \frac{d}{z})^{-2} \to 1 + 2 \frac{d}{z} + \cdots \Theta(\frac{d}{z})^2 \]

\[ (1 + \frac{d}{z})^{-2} \to 1 - 2 \frac{d}{z} + \cdots \Theta(\frac{d}{z})^2 \]

\[ \therefore \mathbf{E} = \frac{Q \hat{\mathbf{z}}}{4\pi \varepsilon \varepsilon_0} \left( \frac{d}{z} \right) = \frac{Qd \hat{\mathbf{z}}}{4\pi \varepsilon \varepsilon_0 z^2} = \frac{Qd \hat{\mathbf{z}}}{4\pi \varepsilon \varepsilon_0 z^2} \cdot \mathbf{E}^0 \]
How to Calculate Electric Fields

Probs. 4.29

- Spherical shell, outer rad = b, inner rad = a
- Volume charge density in shell $p_v = -\frac{\rho_0}{R^2}$, $a \leq R \leq b$

Find $\vec{D} = \epsilon \vec{E}$ everywhere.
How to Calculate Electric Fields

\[ \mathbf{E} = -\nabla V \]

Prob. 4. 36

What is \( \mathbf{E} \)?
How to Calculate Electric Fields

Prob. 4.36
\[ \mathbf{E} = -\nabla V \]

What is \( E^2 \)?

\[ \mathbf{E} = -\nabla V / \left[ +10 \text{V/m} \right] \]

Note: \(-10 \text{V/m}\)

\[ \frac{-30}{3} = -10 \text{V/m} \]
The dielectric strength $E_{ds}$ is the largest magnitude of $E$ that the material can sustain without breakdown.

**Table 4-2:** Relative permittivity (dielectric constant) and dielectric strength of common materials.

<table>
<thead>
<tr>
<th>Material</th>
<th>Relative Permittivity, $\varepsilon_r$</th>
<th>Dielectric Strength, $E_{ds}$ (MV/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air (at sea level)</td>
<td>1.0006</td>
<td>3</td>
</tr>
<tr>
<td>Petroleum oil</td>
<td>2.1</td>
<td>12</td>
</tr>
<tr>
<td>Polystyrene</td>
<td>2.6</td>
<td>20</td>
</tr>
<tr>
<td>Glass</td>
<td>4.5–10</td>
<td>25–40</td>
</tr>
<tr>
<td>Quartz</td>
<td>3.8–5</td>
<td>30</td>
</tr>
<tr>
<td>Bakelite</td>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>Mica</td>
<td>5.4–6</td>
<td>200</td>
</tr>
</tbody>
</table>

$\varepsilon = \varepsilon_r \varepsilon_0$ and $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m.
Dielectric Breakdown

$E_{ds} = \text{dielectric breakdown field.}$

the largest $E$ that can be applied to a dielectric before it turns into a conductor (electrons literally pulled off the atom, a molecule and becomes "free")

Examples:
- air $\rightarrow$ 3 MV/m (30 kV/cm)
- oil $\rightarrow$ 12 MV/m (HV tanks sometimes encased in oil)
- glass $\rightarrow$ 25-40 MV/m
- mica $\rightarrow$ 200 MV/m

$E = \frac{V}{d}$

if $E > E_{ds}$ $\rightarrow$ breakdown

irradiation
lightning

Same effect in integrated circuits
Dielectric Breakdown

big problem in shrinking circuits is DIELECTRIC BREAKDOWN.

as we get thinner and thinner insulators,
we get dielectric breakdown with small voltages.

Example: suppose glass is the insulator.
with \( E_r = 10 \)  \( E_{DS} = 25 \text{MV/m} \)

\[
V_{DS} \sim E_{DS} \times d = 25 \times 10^6 \times \frac{V}{m} (10^{-6} \text{m}) = 25 \text{volts}
\]

\( \therefore \text{at } d = 1 \text{mm gap.} \)

but at \( d = 100 \text{nm} = 0.1 \mu \Rightarrow V_{DS} = 2.5 \text{volts} \)
and at \( d = 10 \text{nm} = 0.01 \mu \Rightarrow V_{DS} = 0.25 \text{volts} \)

so a constant push for higher \( E_r \) materials.

Note: for mica \( E_{DS} = 200 \text{V/m} (\sim 10 \times \text{glass}) \)
so we need increase \( V_{DS} \) by \( \sim 10 \times \)
BUT — mica is unreliable for fabrication.
Dielectric Breakdown.

Geometry also affects breakdown voltage.

E.g., for a parallel plate capacitor, \( V = \frac{E}{d}d \)

What happens if there is a "sharp" point on the surface?

One gets very high \( E \) fields around sharp points.

Remember \( E \propto \frac{1}{r^2} \) from point charge.

And using Gauss' law, you can show that \( E \propto \frac{1}{R^2} \) for a conducting sphere of radius \( R \) at surface.

If \( R << d \) can exceed dielectric breakdown even though \( d \) is large enough on its own.

Process called "Field emission"

Easy to get fields \( > MV/m \)
Resistance

Longitudinal Resistor

\[ V = V_1 - V_2 = - \int_{x_2}^{x_1} \mathbf{E} \cdot d\mathbf{l} \]

\[ = - \int_{x_2}^{x_1} \hat{\mathbf{x}} E_x \cdot \hat{\mathbf{x}} \, dl = E_x l \quad (V). \quad (4.68) \]

Using Eq. (4.63), the current flowing through the cross section \( A \) at \( x_2 \) is

\[ I = \int_A \mathbf{J} \cdot d\mathbf{s} = \int_A \sigma \mathbf{E} \cdot d\mathbf{s} = \sigma E_x A \quad (A). \quad (4.69) \]

From \( R = \frac{V}{I} \), the ratio of Eq. (4.68) to Eq. (4.69) gives

\[ R = \frac{I}{\sigma A} \quad (\Omega). \quad (4.70) \]

For any conductor:

\[ R = \frac{V}{I} = \frac{- \int_{l} \mathbf{E} \cdot d\mathbf{l}}{- \int_{l} \mathbf{J} \cdot d\mathbf{s}} = \frac{- \int_{l} \mathbf{E} \cdot d\mathbf{l}}{- \int_{s} \sigma \mathbf{E} \cdot d\mathbf{s}}. \]

\[ G = 1/R \]
Boundary Conditions

Figure 4-18: Interface between two dielectric media.

\[ E_{lt} = E_{2t} \quad (V/m). \quad (4.90) \]

\[ \hat{n}_2 \cdot (D_1 - D_2) = \rho_s \quad (C/m^2). \]

\[ \frac{D_{lt}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}. \quad (4.91) \]

\[ D_{ln} - D_{2n} = \rho_s \quad (C/m^2). \quad (4.94) \]

The normal component of \( D \) changes abruptly at a charged boundary between two different media in an amount equal to the surface charge density.
Summary of Boundary Conditions

Table 4-3: Boundary conditions for the electric fields.

<table>
<thead>
<tr>
<th>Field Component</th>
<th>Any Two Media</th>
<th>Medium 1 Dielectric $\varepsilon_1$</th>
<th>Medium 2 Conductor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tangential $E$</td>
<td>$E_{1t} = E_{2t}$</td>
<td>$E_{1t} = E_{2t} = 0$</td>
<td></td>
</tr>
<tr>
<td>Tangential $D$</td>
<td>$D_{1t}/\varepsilon_1 = D_{2t}/\varepsilon_2$</td>
<td>$D_{1t} = D_{2t} = 0$</td>
<td></td>
</tr>
<tr>
<td>Normal $E$</td>
<td>$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$</td>
<td>$E_{1n} = \rho_s/\varepsilon_1$</td>
<td>$E_{2n} = 0$</td>
</tr>
<tr>
<td>Normal $D$</td>
<td>$D_{1n} - D_{2n} = \rho_s$</td>
<td>$D_{1n} = \rho_s$</td>
<td>$D_{2n} = 0$</td>
</tr>
</tbody>
</table>

Notes: (1) $\rho_s$ is the surface charge density at the boundary; (2) normal components of $E_1$, $D_1$, $E_2$, and $D_2$ are along $\hat{n}_2$, the outward normal unit vector of medium 2.

Remember $E = 0$ in a good conductor.
Chapter 5 Relationships

Maxwell’s Magnetostatics Equations

Gauss’s Law for Magnetism
\[ \nabla \cdot \mathbf{B} = 0 \quad \leftrightarrow \quad \oint_{\mathcal{S}} \mathbf{B} \cdot ds = 0 \]

Ampère’s Law
\[ \nabla \times \mathbf{H} = \mathbf{J} \quad \leftrightarrow \quad \oint_{\mathcal{C}} \mathbf{H} \cdot d\ell = I \]

Lorentz Force on Charge \( q \)
\[ \mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \]

Magnetic Force on Wire
\[ \mathbf{F}_m = I \oint_{\mathcal{C}} d\ell \times \mathbf{B} \quad (N) \]

Magnetic Torque on Loop
\[ \mathbf{T} = \mathbf{m} \times \mathbf{B} \quad (N\cdot m) \]
\[ \mathbf{m} = \hat{\mathbf{n}} \mathcal{N} I A \quad (A\cdot m^2) \]

Biot–Savart Law
\[ \mathbf{H} = \frac{I}{4\pi} \oint_{\mathcal{C}} \frac{d\ell \times \hat{\mathbf{R}}}{R^2} \quad (A/m) \]

Magnetic Field

Infinitely Long Wire
\[ \mathbf{B} = \hat{\phi} \frac{\mu_0 I}{2\pi r} \quad (\text{Wb/m}^2) \]

Circular Loop
\[ \mathbf{H} = \hat{\mathbf{z}} \frac{Ia^2}{2(a^2 + z^2)\sqrt{2}} \quad (\text{A/m}) \]

Solenoid
\[ \mathbf{B} \simeq \hat{\mathbf{z}} \mu_n I = \frac{\hat{\mathbf{z}} \mu N I}{l} \quad (\text{Wb/m}^2) \]

Vector Magnetic Potential
\[ \mathbf{B} = \nabla \times \mathbf{A} \quad (\text{Wb/m}^2) \]

Vector Poisson’s Equation
\[ \nabla^2 \mathbf{A} = -\mu \mathbf{J} \]

Inductance
\[ L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{1}{I} \oint_{\mathcal{S}} \mathbf{B} \cdot ds \quad (H) \]

Magnetic Energy Density
\[ w_m = \frac{1}{2} \mu H^2 \quad (J/m^3) \]
Summary

Chapter 6 Relationships

Faraday’s Law
\[ V_{\text{emf}} = -\frac{d\Phi}{dt} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} = V_{\text{emf}}^{\text{tr}} + V_{\text{emf}}^{\text{in}} \]

Transformer
\[ V_{\text{emf}}^{\text{tr}} = -N \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} \quad (N \text{ loops}) \]

Motional
\[ V_{\text{emf}}^{\text{in}} = \oint_{\mathcal{C}} (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \]

Charge-Current Continuity
\[ \nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \]

EM Potentials
\[ \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \]
\[ \mathbf{B} = \nabla \times \mathbf{A} \]

Current Density
Conduction
\[ \mathbf{J}_c = \sigma \mathbf{E} \]

Displacement
\[ \mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t} \]

Conductor Charge Dissipation
\[ \rho_v(t) = \rho_{vo} e^{-(\sigma/\varepsilon)t} = \rho_{vo} e^{-t/\tau} \]
**Chapter 7 Relationships**

**Complex Permittivity**
\[
\varepsilon_c = \varepsilon' - j\varepsilon''
\]
\[
\varepsilon' = \varepsilon
\]
\[
\varepsilon'' = \frac{\sigma}{\omega}
\]

**Lossless Medium**
\[
k = \omega \sqrt{\mu \varepsilon}
\]
\[
\eta = \sqrt{\frac{\mu}{\varepsilon}} \quad (\Omega)
\]
\[
u_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu \varepsilon}} \quad (\text{m/s})
\]
\[
\lambda = \frac{2\pi}{k} = \frac{u_p}{f} \quad (\text{m})
\]

**Wave Polarization**
\[
\mathbf{H} = \frac{1}{\eta} \mathbf{k} \times \mathbf{E}
\]
\[
\mathbf{E} = -\eta \mathbf{k} \times \mathbf{H}
\]

**Maxwell’s Equations for Time-Harmonic Fields**
\[
\nabla \cdot \mathbf{E} = 0
\]
\[
\nabla \times \mathbf{E} = -j\omega \mu \mathbf{H}
\]
\[
\nabla \cdot \mathbf{H} = 0
\]
\[
\nabla \times \mathbf{H} = j\omega \varepsilon_c \mathbf{E}
\]

**Lossy Medium**
\[
\alpha = \omega \left\{ \frac{\mu \varepsilon'}{2} \left[ \sqrt{1 + \left( \frac{\varepsilon''}{\varepsilon'} \right)^2} - 1 \right] \right\}^{1/2} \quad (\text{Np/m})
\]
\[
\beta = \omega \left\{ \frac{\mu \varepsilon'}{2} \left[ \sqrt{1 + \left( \frac{\varepsilon''}{\varepsilon'} \right)^2} + 1 \right] \right\}^{1/2} \quad (\text{rad/m})
\]
\[
\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon'}} \left( 1 - j \frac{\varepsilon''}{\varepsilon'} \right)^{-1/2} \quad (\Omega)
\]
\[
\delta_c = \frac{1}{\alpha} \quad (\text{m})
\]

**Power Density**
\[
S_{av} = \frac{1}{2} \Re \left[ \mathbf{E} \times \mathbf{H}^* \right] \quad (\text{W/m}^2)
\]
Dielectric Materials

Non-polar

Figure 4-16: In the absence of an external electric field $E$, the center of the electron cloud is co-located with the center of the nucleus, but when a field is applied, the two centers are separated by a distance $d$.

Figure 4-17: A dielectric medium polarized by an external electric field $E$. 
Polarization Field

\[ \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \]

\( \mathbf{P} = \text{electric flux density induced by} \ \mathbf{E} \)

\[ \mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}, \quad (4.84) \]

where \( \chi_e \) is called the \textit{electric susceptibility} of the material. Inserting Eq. (4.84) into Eq. (4.83), we have

\[ \mathbf{D} = \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi_e \mathbf{E} \]
\[ = \varepsilon_0 (1 + \chi_e) \mathbf{E} = \varepsilon \mathbf{E}, \quad (4.85) \]
Conductors

Net electric field inside a conductor is zero

Figure 4-20: When a conducting slab is placed in an external electric field \( E_1 \), charges that accumulate on the conductor surfaces induce an internal electric field \( E_i = -E_1 \). Consequently, the total field inside the conductor is zero.
Charge within a conductor

\[ \text{why is there an } E \text{ field here} \]
\[ \text{which causes current to flow?} \]

ANS. Not in equilibrium
\[ \text{supplying external "energy"!} \]
Capacitance

When a conductor has excess charge, it distributes the charge on its surface in such a manner as to maintain a zero electric field everywhere within the conductor, thereby ensuring that the electric potential is the same at every point in the conductor.

The *capacitance* of a two-conductor configuration is defined as

$$ C = \frac{Q}{V} \quad \text{(C/V or F),} \quad (4.105) $$

*Figure 4-23:* A dc voltage source connected to a capacitor composed of two conducting bodies.
Capacitance

Since $E_{\text{tan}}$ is continuous across boundaries and $E=0$ inside conductors, the tangential component always vanishes on the surface of conductors.

$E$ always $\perp$. 

\[ E_n = n \cdot E = \frac{\rho_s}{\varepsilon} \text{ at conductor surface} \]

\[ Q = \oint_{S_+} \varepsilon \cdot E \cdot dS = \oint_{S_-} \varepsilon \cdot E \cdot dS \]

\[ V = -\frac{1}{2} \int E \cdot dl \]

- Capacitance $C = \frac{Q}{V} = \frac{\oint_{S_+} \varepsilon \cdot E \cdot dS}{-\frac{1}{2} \int E \cdot dl}$

\[ \text{NOTE: since } E \text{ appears on top and bottom, } C \text{ is indep of } E \text{. only depends on material and geometry.} \]
Capacitance

If material between conductors not perfect dielectric, i.e. $\sigma \neq 0$, then current can flow between the two conductors → a "resistance".

From last lecture, we had resistance for an arbitrary shape was

\[ R = \frac{V}{I} = -\frac{\int E \cdot d\vec{l}}{\oint \phi_0 E \cdot d\vec{s}} = R \]

If medium has uniform $\sigma$, $\varepsilon$ then

\[ RC = \frac{-\oint E \cdot d\vec{l}}{\oint \phi_0 E \cdot d\vec{s}} \cdot \oint \phi_0 E \cdot d\vec{s} = \frac{\varepsilon}{\sigma} = RC \]

Thus, we can find $R$ if $C$ is known

a vice versa principle of lab 2
Electrostatic Potential Energy

Electrostatic potential energy density (Joules/volume)

\[ w_e = \frac{W_e}{V} = \frac{1}{2} \varepsilon E^2 \quad (\text{J/m}^3). \]

Energy stored in a capacitor

\[ W_e = \frac{1}{2} C V^2 \quad (\text{J}). \]

Total electrostatic energy stored in a volume

\[ W_e = \frac{1}{2} \int_V \varepsilon E^2 \, dV \quad (\text{J}) \]
### Chapter 5 Relationships

#### Maxwell’s Magnetostatics Equations

**Gauss’s Law for Magnetism**
\[ \nabla \cdot \mathbf{B} = 0 \quad \leftrightarrow \quad \oint_S \mathbf{B} \cdot d\mathbf{s} = 0 \]

**Ampère’s Law**
\[ \nabla \times \mathbf{H} = \mathbf{J} \quad \leftrightarrow \quad \oint_C \mathbf{H} \cdot d\mathbf{l} = I \]

#### Magnetic Field

**Infinitely Long Wire**
\[ \mathbf{B} = \frac{\hat{\mathbf{z}} \mu_0 I}{2\pi r} \quad \text{(Wb/m}^2\text{)} \]

**Circular Loop**
\[ \mathbf{H} = \hat{\mathbf{z}} \frac{1a^2}{2(a^2+z^2)^{3/2}} \quad \text{(A/m)} \]

**Solenoid**
\[ \mathbf{B} \propto \hat{\mathbf{z}} \mu_0 n I = \frac{\hat{\mathbf{z}} / \mu_0 N I}{l} \quad \text{(Wb/m}^2\text{)} \]

#### Lorentz Force on Charge \( q \)
\[ \mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \]

#### Magnetic Force on Wire
\[ F_m = I \oint_C d\mathbf{l} \times \mathbf{B} \quad \text{(N)} \]

#### Magnetic Torque on Loop
\[ \mathbf{T} = m \times \mathbf{B} \quad \text{(N} \cdot \text{m)} \]
\[ m = \hat{\mathbf{n}} N I A \quad \text{(A} \cdot \text{m}^2\text{)} \]

#### Biot–Savart Law
\[ \mathbf{H} = \frac{I}{4\pi} \int_C \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \quad \text{(A/m)} \]

#### Vector Magnetic Potential
\[ \mathbf{B} = \nabla \times \mathbf{A} \quad \text{(Wb/m}^2\text{)} \]

#### Vector Poisson’s Equation
\[ \nabla^2 \mathbf{A} = -\mu \mathbf{J} \]

#### Inductance
\[ L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{1}{l} \oint_S \mathbf{B} \cdot d\mathbf{s} \quad \text{(H)} \]

#### Magnetic Energy Density
\[ w_m = \frac{1}{2} \mu H^2 \quad \text{(J/m}^3\text{)} \]
magnetic forces

\[ \vec{F} = q \vec{E} \] called \( \vec{F} \) coulomb force

\[ \text{mag flux density}, \vec{B} \]

mag field is defined as in terms of a "magnetic force" acting on a moving charged particle passing thru that pt.

based upon exp. measurements

\[ \vec{F}_m = q \vec{u} \times \vec{B} \] Lenz\'s Force (not as easy to invert \( \vec{B} \) in terms of \( \vec{F} \) as with elec field)

\( \vec{B} \) in newtons/amp/m\_sqm = Tesla

 NOTE: mag force acts \( \pm \) direction of motion and direction of mag field!
**NOTE**:

**Magnetic Force Acts** + direction of motion and - direction of mag field!

- Right hand rule -

\[ |\mathbf{F}_m| = q\mathbf{u} \cdot \mathbf{B} \sin \theta \]

\[ \mathbf{u} \parallel \mathbf{B} \] or \[ \mathbf{u} \perp \mathbf{B} \]

*NOTE*:

- Magnetic force only acts on moving particles ("currents")

**NOTE**:

- Difference between Electric and Magnetic forces:
  - Electric force:
    \[ dW_{elec} = \mathbf{F}_{elec} \cdot d\mathbf{r} \]
  - Magnetic force:
    \[ dW_{mag} = \mathbf{F}_m \cdot d\mathbf{r} = (\mathbf{F}_m \cdot \mathbf{u} dt) = 0 \]

\( \text{since} \quad \mathbf{F}_m \perp \mathbf{u} \quad \text{in a mag field!} \)
If an elec charge particle is moving in both \( E \) \( B \) field force on it is:
\[
\vec{F} = \vec{F}_e + \vec{F}_m = q\vec{E} + q\vec{v} \times \vec{B}
\]

\[
\vec{F}_{EB} = q(\vec{E} + \vec{v} \times \vec{B})
\]

Lorentz force

Important differences between elec \& mag forces:

1. \( \vec{F}_e \parallel \vec{E} \)
   \( \vec{F}_B \perp \vec{B} \)

2. \( \vec{F}_e \) acts on \( q \) moving or not
   \( \vec{F}_B \) only acts on moving \( q \)

3. (we will prove)
   work done on moving charged particle with \( \vec{E} \)
   no work done on moving charged particle with \( \vec{B} \)
   prove \( \int dw = \vec{E} \cdot d\vec{l} \)
   \( \vec{E} \cdot d\vec{l} = (q\vec{E}) \cdot d\vec{l} \)

   may \( \int dw = (q\vec{v} \times \vec{B}) \cdot d\vec{l} \) but \( d\vec{l} = \vec{v} dt \)

   \[
   dw = q(\vec{v} \times \vec{B}) \cdot (\vec{v} dt)
   \]

   \[
   \int \vec{v} \times \vec{B} \cdot \vec{v} dt
   \]

   \[
   dw = 0!
   \]

**Note**

\( \vec{E} \) mag field cannot change the energy of a charged particle; can only change direction of motion (ie momentum)
Magnetic Force on a Current Element

Differential force $dF_m$ on a differential current $I \, dl$:

$$dF_m = I \, dl \times B \quad (\text{N}). \quad (5.9)$$

For a closed circuit of contour $C$ carrying a current $I$, the total magnetic force is

$$F_m = I \oint_C dl \times B \quad (\text{N}). \quad (5.10)$$

If the closed wire shown in Fig. 5-3(a) resides in a uniform external magnetic field $B$, then $B$ can be taken outside the integral in Eq. (5.10), in which case

$$F_m = I \left( \oint_C dl \right) \times B = 0. \quad (5.11)$$

This result, which is a consequence of the fact that the vector sum of the infinitesimal vectors $dl$ over a closed path equals zero, states that the total magnetic force on any closed current loop in a uniform magnetic field is zero.

Figure 5-2: When a slightly flexible vertical wire is placed in a magnetic field directed into the page (as denoted by the crosses), it is (a) not deflected when the current through it is zero, (b) deflected to the left when $I$ is upward, and (c) deflected to the right when $I$ is downward.
Magnetic Torque on Current Loop

\[ \mathbf{F}_1 = I (\hat{\mathbf{y}} b) \times (\hat{\mathbf{x}} B_0) = \hat{\mathbf{z}} I l b B_0, \]

\[ \mathbf{F}_3 = I (\hat{\mathbf{y}} b) \times (\hat{\mathbf{x}} B_0) = -\hat{\mathbf{z}} I l b B_0. \]

No forces on arms 2 and 4 (because \( I \) and \( B \) are parallel, or anti-parallel)

**Magnetic torque:**

\[ \mathbf{T} = \mathbf{d}_1 \times \mathbf{F}_1 + \mathbf{d}_3 \times \mathbf{F}_3 \]

\[ = \left( -\hat{\mathbf{x}} \frac{a}{2} \right) \times (\hat{\mathbf{z}} I l b B_0) + \left( \hat{\mathbf{x}} \frac{a}{2} \right) \times (-\hat{\mathbf{z}} I l b B_0) \]

\[ = \hat{\mathbf{y}} I a b B_0 = \hat{\mathbf{y}} I A B_0, \]

Area of Loop

---

**Figure 5.6:** Rectangular loop pivoted along the \( y \)-axis: (a) front view and (b) bottom view. The combination of forces \( \mathbf{F}_1 \) and \( \mathbf{F}_3 \) on the loop generates a torque that tends to rotate the loop in a clockwise direction as shown in (b).
Biot-Savart Law

Magnetic field induced by a differential current:

\[ dH = \frac{I}{4\pi} \frac{dl \times \hat{R}}{R^2} \] (A/m)

For the entire length:

\[ H = \frac{I}{4\pi} \int\limits_{l} dl \times \hat{R} \] (A/m), \hspace{1cm} (5.22)

where \( l \) is the line path along which \( I \) exists.

**Figure 5-8:** Magnetic field \( dH \) generated by a current element \( I \, dl \). The direction of the field induced at point \( P \) is opposite to that induced at point \( P' \).
Biot-Savart Law. 2.

calculatation of $\vec{H}$ due to a current carrying wire -

the geometry:

$\vec{H}$

$\vec{dl} = \hat{z} dz$

$\vec{R} = r^2 + z^2$ a vector quantity

$\therefore \vec{dl} \times \vec{R} = (\hat{z} dz) \times (r^2 + z^2 \hat{z})$

$= dz (r^2 \hat{x} + z^2 \hat{z}) = dz r^2 \hat{x}$

$\therefore \frac{\vec{dl} \times \vec{R}}{4\pi} = \frac{dz r^2 \hat{x}}{4\pi}$

$\therefore \frac{dH}{4\pi} = \frac{I \vec{dl} \times \vec{R}}{4\pi} = \frac{I r dz \hat{\phi}}{4\pi (r^2 + z^2)^{3/2}} = \frac{d\vec{H}}{4\pi}$
Biot–Savart Law. 3

\[
\vec{H} \text{ due to current carrying wire}
\]

\[
\frac{d\vec{H}}{4\pi} = \frac{I}{(r^2+z^2)^{3/2}} d\theta
\]

\[
\text{two ways to integrate } d\vec{H}
\]

1. Convert to angles, since \(d\theta\) a function of \(\theta\).
   - Remember \(\theta\) is between \(d\theta\) and \(R\).
   - \(r/\theta = \sin\theta\)
   - \(R = r \csc \theta\)
   - \(z = r \cot \theta \Rightarrow dz = r d(\cot \theta) = -r \csc^2 \theta d\theta\).

\[
\vec{H} = \frac{I}{4\pi} \int_{\theta_1}^{\theta_2} \frac{r \csc^2 \theta d\theta}{(r^2+z^2)^{3/2}} = \frac{I}{4\pi} \int_{\theta_1}^{\theta_2} \frac{r \csc^2 \theta d\theta}{r^2 \csc^2 \theta}
\]

\[
\vec{H} = \frac{I}{4\pi} \int_{\theta_1}^{\theta_2} \frac{\csc^2 \theta d\theta}{(r^2+z^2)^{3/2}} = \frac{I}{4\pi} \int_{\theta_1}^{\theta_2} \frac{\csc^2 \theta d\theta}{r^2}
\]

\[
\vec{H} = \frac{I}{4\pi} \left[ \csc \theta \right]_{\theta_1}^{\theta_2} + \left( \cot \theta_1 - \cot \theta_2 \right)
\]

For \(l \to \infty\), \(l = l_1 + l_2\) — infinitely long wire.

Then \(\theta_2 \to \pi\), \(\theta_1 \to 0\), \(\cot \theta_1 - \cot \theta_2 = z\)

\[
\vec{H} = \frac{I}{2\pi l} \text{ field circular around the wire.}
\]
2. 2nd way to integrate \( \int \frac{dr}{r} \):

\[ \vec{H} = \frac{I}{4\pi} \int \frac{dr \vec{z} \cdot \hat{\mathbf{e}}}{(r^2 + z^2)^{3/2}} \]

\[ = \frac{I}{4\pi} \int \frac{z \, \vec{z}}{(r^2 + z^2)^{3/2}} \Bigg|_{-l_1}^{+l_2} \]

\[ = \frac{I}{4\pi} \bigg[ \frac{z}{r^2 + z^2} \bigg] \bigg|_{-l_1}^{+l_2} \]

\[ \vec{H} = \frac{I}{4\pi} \int \bigg[ \frac{l_2}{r^2 + l_2^2} + \frac{l_1}{r^2 + l_1^2} \bigg] \]

\[ \text{but} \quad \frac{l_2}{r^2 + l_2^2} = \cos(\pi - \theta_2) = -\cos \theta_2 \]

\[ \frac{l_1}{r^2 + l_1^2} = \cos \theta_1 \]

\[ \therefore \quad \vec{H} = \frac{I}{4\pi} \int \bigg[ \cos \theta_1 - \cos \theta_2 \bigg] \text{ as before.} \]
Magnetic Field of Long Conductor

\[ B = \Phi \frac{\mu_0 I}{2\pi r} \]  
(infinitely long wire).
Example 5-3: Magnetic Field of a Loop

Magnitude of field due to \( dl \) is

\[
dH = \frac{I}{4\pi R^2} |dl \times \hat{R}| = \frac{I}{4\pi (a^2 + z^2)} dl
\]

\( dH \) is in the \( r-z \) plane, and therefore it has components \( dH_r \) and \( dH_z \)

\( z \)-components of the magnetic fields due to \( dl \) and \( dl' \) add because they are in the same direction, but their \( r \)-components cancel

Hence for element \( dl \):

\[
dH = \hat{z} dH_z = \hat{z} dH \cos \theta = \hat{z} \frac{I \cos \theta}{4\pi (a^2 + z^2)} \ dl
\]

Figure 5-12: Circular loop carrying a current \( I \) (Example 5-3).
Magnetic Dipole

Because a circular loop exhibits a magnetic field pattern similar to the electric field of an electric dipole, it is called a magnetic dipole.
Ampère’s Law

\[ \nabla \times \mathbf{H} = \mathbf{J} \iff \oint_{C} \mathbf{H} \cdot d\ell = I \]

The sign convention for the direction of the contour path \( C \) in Ampère’s law is taken so that \( I \) and \( \mathbf{H} \) satisfy the right-hand rule defined earlier in connection with the Biot–Savart law. That is, if the direction of \( I \) is aligned with the direction of the thumb of the right hand, then the direction of the contour \( C \) should be chosen along that of the other four fingers.

**Figure 5-16**: Ampère’s law states that the line integral of \( \mathbf{H} \) around a closed contour \( C \) is equal to the current traversing the surface bounded by the contour. This is true for contours (a) and (b), but the line integral of \( \mathbf{H} \) is zero for the contour in (c) because the current \( I \) (denoted by the symbol \( \bigcirc \)) is not enclosed by the contour \( C \).
Internal Magnetic Field of Long Conductor

For \( r < a \)

\[
\oint_{C_1} \mathbf{H}_1 \cdot d\mathbf{l}_1 = I_1,
\]

\[
\oint_{C_1} \mathbf{H}_1 \cdot d\mathbf{l}_1 = \int_0^{2\pi} H_1(\hat{\phi} \cdot \hat{\phi})r_1 \, d\phi = 2\pi r_1 H_1.
\]

The current \( I_1 \) flowing through the area enclosed by \( C_1 \) is equal to the total current \( I \) multiplied by the ratio of the area enclosed by \( C_1 \) to the total cross-sectional area of the wire:

\[
I_1 = \left( \frac{\pi r_1^2}{\pi a^2} \right) I = \left( \frac{r_1}{a} \right)^2 I.
\]

Equating both sides of Eq. (5.48) and then solving for \( \mathbf{H}_1 \) yields

\[
\mathbf{H}_1 = \hat{\phi} H_1 = \hat{\phi} \frac{r_1}{2\pi a^2} I \quad \text{(for} \ r_1 \leq a) \quad \text{(5.49)}
\]
External Magnetic Field of Long Conductor

For \( r > a \)

(b) For \( r = r_2 \geq a \), we choose path \( C_2 \), which encloses all the current \( I \). Hence, \( H_2 = \hat{\phi} H_2 \), \( d\ell_2 = \hat{\phi} r_2 \, d\phi \), and

\[
\oint_{C_2} H_2 \cdot d\ell_2 = 2\pi r_2 H_2 = I,
\]

which yields

\[
H_2 = \hat{\phi} H_2 = \hat{\phi} \frac{I}{2\pi r_2} \quad \text{for} \ r_2 \geq a.
\] (5.49b)
Magnetic Properties of Materials

The magnetic behavior of a material is governed by the interaction of the magnetic dipole moments of its atoms with an external magnetic field. The nature of the behavior depends on the crystalline structure of the material and is used as a basis for classifying materials as diamagnetic, paramagnetic, or ferromagnetic.

\[ B = \mu_0 H + \mu_0 M = \mu_0 (H + M) \]

\[ M = \chi_m H \]

\[ B = \mu_0 (H + \chi_m H) = \mu_0 (1 + \chi_m)H, \]

\[ B = \mu H, \]
The magnetic field in the region $S$ between the two conductors is approximately

$$B = \hat{\Phi} \frac{\mu I}{2\pi r}$$

Total magnetic flux through $S$:

$$\Phi = l \int_{a}^{b} B \, dr = l \int_{a}^{b} \frac{\mu I}{2\pi r} \, dr = \frac{\mu I l}{2\pi} \ln \left( \frac{b}{a} \right)$$

Inductance per unit length:

$$L' = \frac{L}{l} = \frac{\Phi}{l I} = \frac{\mu}{2\pi} \ln \left( \frac{b}{a} \right).$$

**Example 5-7: Inductance of Coaxial Cable**

**Figure 5-28:** Cross-sectional view of coaxial transmission line (Example 5-7).
Maxwell’s Equations

In this chapter, we will examine Faraday’s and Ampère’s laws

<table>
<thead>
<tr>
<th>Reference</th>
<th>Differential Form</th>
<th>Integral Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauss’s law</td>
<td>( \nabla \cdot \mathbf{D} = \rho_v )</td>
<td>( \oint_S \mathbf{D} \cdot d\mathbf{s} = Q ) (6.1)</td>
</tr>
<tr>
<td>Faraday’s law</td>
<td>( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} )</td>
<td>( \oint_C \mathbf{E} \cdot d\mathbf{l} = -\oint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} ) (6.2)*</td>
</tr>
<tr>
<td>Gauss’s law for magnetism</td>
<td>( \nabla \cdot \mathbf{B} = 0 )</td>
<td>( \oint_S \mathbf{B} \cdot d\mathbf{s} = 0 ) (6.3)</td>
</tr>
<tr>
<td>Ampère’s law</td>
<td>( \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} )</td>
<td>( \oint_C \mathbf{H} \cdot d\mathbf{l} = \oint_S \left( \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s} ) (6.4)</td>
</tr>
</tbody>
</table>

*For a stationary surface \( S \).
Chapter 6 Relationships

Faraday’s Law
\[ V_{\text{emf}} = -\frac{d\Phi}{dt} = -\frac{d}{dt} \int_S \mathbf{B} \cdot ds = V^{\text{tr}}_{\text{emf}} + V^{\text{in}}_{\text{emf}} \]

Transformer
\[ V^{\text{tr}}_{\text{emf}} = -N \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot ds \quad (N \text{ loops}) \]

Motional
\[ V^{\text{in}}_{\text{emf}} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \]

Charge-Current Continuity
\[ \nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \]

EM Potentials
\[ \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \]
\[ \mathbf{B} = \nabla \times \mathbf{A} \]

Current Density
Conduction \[ \mathbf{J}_c = \sigma \mathbf{E} \]
Displacement \[ \mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t} \]

Conductor Charge Dissipation
\[ \rho_v(t) = \rho_{vo} e^{-(\alpha/\varepsilon)t} = \rho_{vo} e^{-t/\tau} \]
EM Potentials

Faraday's law: \[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

but \[ \vec{B} = \nabla \times \vec{A} \] — the "vector potential"

\[ \nabla \times \vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{A}) \]

\[ \nabla \times (\vec{E} + \frac{\partial \vec{B}}{\partial t}) = 0 \]

Let \[ \vec{E}' = \vec{E} + \frac{\partial \vec{B}}{\partial t} \]

then \[ \nabla \times \vec{E}' = 0 \] as in static case

but with this we can define

\[ \vec{E}' = -\nabla V \]

\[ \therefore \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \]

the electric field in terms of the scalar electric potential

AND

the time-varying magnetic vector potential

Thus if \( V \) and \( \vec{A} \) are known we can get \( \vec{E}' \).
We must take into account that it takes time for the signal to get from source to the “observation” point. That time is \( R' / u_p \), where \( u_p \) = velocity of light.
Faraday’s Law

Electromotive force (voltage) induced by time-varying magnetic flux:

\[ V_{\text{emf}} = -N \frac{d\Phi}{dt} = -N \frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} \] (V)

*Magnetic fields can produce an electric current in a closed loop, but only if the magnetic flux linking the surface area of the loop changes with time. The key to the induction process is change.*
Three types of EMF

1. A time-varying magnetic field linking a stationary loop; the induced emf is then called the *transformer emf*, $V_{\text{emf}}^{\text{tr}}$.

2. A moving loop with a time-varying surface area (relative to the normal component of $\mathbf{B}$) in a static field $\mathbf{B}$; the induced emf is then called the *motional emf*, $V_{\text{emf}}^{\text{m}}$.

3. A moving loop in a time-varying field $\mathbf{B}$.

The total emf is given by

$$V_{\text{emf}} = V_{\text{emf}}^{\text{tr}} + V_{\text{emf}}^{\text{m}}, \quad (6.7)$$
Example of Lenz’s law

CLOSE switch, what is direction of I in upper loop as switch is closed —
Example of Lenz's law

CLOSED switch, what is direction of I in upper loop as switch is closed:

$I_{bottom} = CW$

$\Rightarrow B$ in top loop increasing out of page

$\Rightarrow I$ in top loop produces field to oppose that $\Rightarrow B$ out of page $/ CW$

OPEN switch:

$I_{bottom} = CW$

$\Rightarrow I$ in top loop produces field to oppose that $\Rightarrow B$ out of page $\Rightarrow I$ is $CCW$
Problem 6.13  The circular, conducting, disk shown in P6.13 lies in the $x$–$y$ plane and rotates with uniform angular velocity $\omega$ about the $z$-axis. The disk is of radius $a$ and is present in a uniform magnetic flux density $\mathbf{B} = \hat{z}B_0$. Obtain an expression for the emf induced at the rim relative to the center of the disk.

![Diagram of a rotating disk in a magnetic field](image)

**Figure P6.13:** Rotating circular disk in a magnetic field (Problem 6.13).
Faraday's Homopolar Generator

at radial distance, $r$

velocity of point of rotating disc = $v$

$\vec{v} = q \omega \vec{r}$

$x$ freq. of rot.

about $2 \pi \text{ rpm}$

$\vec{F} = q \vec{A} \times \vec{B}$

Induced EMF = $V = \int_0^a (\hat{\omega} \vec{r} \times \hat{z} \vec{B}) \cdot \vec{dl}$

center to rim (rad = $a$)

but $\vec{dl} = \vec{r} dr$

$V = \int_0^a (\hat{\omega} \vec{r} \times \hat{z} \vec{B}) \cdot \vec{r} dr$

$\hat{\omega} \times \hat{z}$ along $\hat{r}$ direction : $(\hat{\omega} \times \hat{r}) \cdot \vec{r} = 1$

$V = \int_0^a \vec{r} \cdot \vec{B}_0 \vec{r} dr = \frac{1}{2} \vec{B}_0 a^2 = V A$
Summary

**Chapter 6 Relationships**

**Faraday’s Law**

\[ V_{\text{emf}} = -\frac{d\Phi}{dt} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} = V_{\text{emf}}^{\text{tr}} + V_{\text{emf}}^{\text{in}} \]

**Transformer**

\[ V_{\text{emf}}^{\text{tr}} = -N \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} \quad \text{(N loops)} \]

**Motional**

\[ V_{\text{emf}}^{\text{in}} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \]

**Charge-Current Continuity**

\[ \nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \]

**EM Potentials**

\[ \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \]

\[ \mathbf{B} = \nabla \times \mathbf{A} \]

**Current Density**

**Conduction**

\[ \mathbf{J}_c = \sigma \mathbf{E} \]

**Displacement**

\[ \mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t} \]

**Conductor Charge Dissipation**

\[ \rho_v(t) = \rho_{vo} e^{-(\alpha/\varepsilon)t} = \rho_{vo} e^{-t/\tau} \]
### Chapter 7 Relationships

#### Complex Permittivity

\[
\varepsilon_c = \varepsilon' - j\varepsilon'' \\
\varepsilon' = \varepsilon \\
\varepsilon'' = \frac{\sigma}{\omega}
\]

#### Lossless Medium

\[
k = \omega \sqrt{\mu \varepsilon} \\
\eta = \sqrt{\frac{\mu}{\varepsilon}} \quad (\Omega) \\
u_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu \varepsilon}} \quad (\text{m/s}) \\
\lambda = \frac{2\pi}{k} = \frac{u_p}{f} \quad (\text{m})
\]

#### Wave Polarization

\[
\hat{H} = \frac{1}{\eta} \hat{k} \times \hat{E} \\
\hat{E} = -\eta \hat{k} \times \hat{H}
\]

#### Maxwell’s Equations for Time-Harmonic Fields

\[
\nabla \cdot \vec{E} = 0 \\
\nabla \times \vec{E} = -j\omega \mu \vec{H} \\
\nabla \cdot \vec{H} = 0 \\
\nabla \times \vec{H} = j\omega \varepsilon_c \vec{E}
\]

#### Lossy Medium

\[
\alpha = \omega \left\{ \frac{\mu \varepsilon'}{2} \left[ \sqrt{1 + \left( \frac{\varepsilon''}{\varepsilon'} \right)^2} - 1 \right] \right\}^{1/2} \quad (\text{Np/m}) \\
\beta = \omega \left\{ \frac{\mu \varepsilon'}{2} \left[ \sqrt{1 + \left( \frac{\varepsilon''}{\varepsilon'} \right)^2} + 1 \right] \right\}^{1/2} \quad (\text{rad/m}) \\
\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon'}} \left( 1 - j\frac{\varepsilon''}{\varepsilon'} \right)^{-1/2} \quad (\Omega) \\
\delta_c = \frac{1}{\alpha} \quad (\text{m})
\]

#### Power Density

\[
S_{av} = \frac{1}{2} \Re \left[ \hat{E} \times \hat{H}^* \right] \quad (\text{W/m}^2)
\]
Structure of final exam

10 problems

Several short problem sets/
one liners, quick
calculations, concepts

Just try to understand the concepts.
EE 135 Cheat Sheet Information

The following tables and charts will be given to you for the final exam:

Chapter relationships that appear at the end of each chapter for chapters 1-7. Including the tables.

The front inside cover of the text.
The back inside cover of the text

Table 3.1, 3.2