Lecture 2

Electromagnetic Optics - Wave Nature of Light

• Light waves in a homogeneous medium
• Refractive Index
• Group velocity and group index
• Magnetic field, irradiance, and Poynting vector
• Snell's Law and TIR
• Fresnel Equations
Limits of Geometrical Optics

• Ray Optics:
  • Snell’s Law
  • Imaging

• Questions:
  • How much light is reflected / refracted?
  • What is the brightness of the image?
  • Diffraction
Electromagnetic Optics

- Maxwell’s Equations (MKS units):
  \[
  \begin{align*}
  \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \\
  \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} &= \mathbf{J} \\
  \nabla \cdot \mathbf{D} &= \rho \\
  \nabla \cdot \mathbf{B} &= 0
  \end{align*}
  \]
  \(\mathbf{E}\) = electric field vector
  \(\mathbf{H}\) = magnetic field vector
  \(\mathbf{D}\) = electric displacement
  \(\mathbf{B}\) = magnetic induction
  \(\rho\) = electric charge density
  \(\mathbf{J}\) = current density

- Constitutive Equations (or material equations):
  \[
  \begin{align*}
  \mathbf{D} &= \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P} \\
  \mathbf{B} &= \mu \mathbf{H} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M}
  \end{align*}
  \]
  \(\varepsilon\) = dielectric tensor (or permittivity tensor)
  \(\mu\) = permeability tensor
  \(\varepsilon_0\) = permittivity of vacuum
  \(\mu_0\) = permeability of vacuum
  \(\mathbf{P}\) = electric polarization
  \(\mathbf{M}\) = magnetic polarizations
Wave Equations and Monochromatic Plane Waves:

\[ \nabla \times (\nabla \times \vec{E}) + \frac{\partial}{\partial t} \nabla \times \vec{B} = 0 \]

\[ \nabla \times (\nabla \times \vec{E}) = \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} \]

\[ \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} + \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \]

Similarly,

\[ \nabla^2 \vec{B} - \mu \varepsilon \frac{\partial^2 \vec{B}}{\partial t^2} = 0 \]
Plane waves:

\[ \vec{E} = \vec{E}_0 \, e^{j(k \cdot \hat{r} - \omega t)} \]

Phase velocity

\[ \vec{B} = \vec{B}_0 \, e^{j(k \cdot \hat{r} - \omega t)} \]

\[ \nu = \frac{\omega}{k} = \frac{1}{\sqrt{\mu \varepsilon}} \]

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \vec{j} \hat{k} \times \vec{E} = j \omega \vec{B} \]

\[ \nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t} \Rightarrow \vec{j} \hat{k} \times \vec{B} = \mu \varepsilon (-j \omega) \vec{E} \]

\[ \varepsilon_0 = 8.85 \times 10^{-12} \text{farad/m} \]

\[ \mu_0 = 4\pi \times 10^{-7} \text{newton} \times \text{ampere}^{-1} \]

\[ c = 3 \times 10^8 \text{m/sec} \]

\[ \hat{\vec{E}}, \hat{\vec{B}}, \hat{\vec{k}} \text{ are mutually orthogonal to one another} \]

\[ \nu = \frac{1}{\sqrt{\mu \varepsilon}} = \frac{1}{\sqrt{\mu_0 \varepsilon_0} \sqrt{\mu \varepsilon}} = \frac{c}{n} \text{ refractive index} \]
An electromagnetic wave is a travelling wave which has time varying electric and magnetic fields which are perpendicular to each other and the direction of propagation, \( z \).

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**Plane Wave**

\[
E_x = E_0 \cos(\omega t - k z + \phi_0)
\]

- \( E_0 \) = amplitude
- \( \omega \) = angular frequency
- \( t \) = time
- \( k \) = propagation constant (wave number)
- \( \phi_0 \) = phase constant
- \( z \) = position
- \( \lambda \) = wavelength
A plane EM wave travelling along $z$, has the same $E_x$ (or $B_y$) at any point in a given $xy$ plane. All electric field vectors in a given $xy$ plane are therefore in phase. The $xy$ planes are of infinite extent in the $x$ and $y$ directions.

\[ E_x = E_0 \sin(\omega t - kz) \]

Wavefront
\[ \phi = \omega t - kz + \phi_0 = \text{constant} \]

Phase velocity
\[ v = \frac{dz}{dt} = \frac{\omega}{k} = \frac{c}{n} \]
A travelling plane EM wave along a direction $\mathbf{k}$

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Wave fronts
(constant phase surfaces)

A perfect plane wave
(a)

A perfect spherical wave
(b)

A divergent beam
(c)

Examples of possible EM waves

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Wave equation:

\[ \nabla^2 E = \varepsilon \mu \frac{\partial^2 E}{\partial t^2} \quad \left( \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = \varepsilon_0 \varepsilon_r \mu_0 \frac{\partial^2 E}{\partial t^2} \right) \]

Spherical wave

\[ E = \frac{A}{r} \cos (\omega t - kr) \]
(a) Wavefronts of a Gaussian light beam. (b) Light intensity across beam cross section. (c) Light irradiance (intensity) vs. radial distance $r$ from beam axis ($z$).

Gaussian beam

$2w = \text{beam diameter}$

$85\%$ of the beam power is within the $\pi w^2$ area.

Beam divergence

$\theta = \frac{4\lambda}{\pi(2w_o)}$

<table>
<thead>
<tr>
<th>Example 1.1.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda = 633 \text{ nm}$ for He-Ne, spot size $10 \text{ mm}$,</td>
</tr>
<tr>
<td>$2\theta = \frac{4\lambda}{\pi(2w_o)} = \frac{4 \times 633 \times 10^{-9} \text{ m}}{\pi(10 \times 10^{-3} \text{ m})} = 8.06 \times 10^{-5} \text{ rad} = 0.0046^\circ$</td>
</tr>
</tbody>
</table>
Refractive Index

- Interaction between EM field and molecules —
  Polarization mechanism slows down the EM wave

- Phase velocity
  \[ v = \frac{1}{\sqrt{\varepsilon_r \varepsilon_0 \mu_0}} \]
  \[ c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3 \times 10^8 \text{ m/s} \]

- Refractive index
  \[ n = \frac{c}{v} = \sqrt{\varepsilon_r} \]

- \( k_{\text{medium}} = n k \)

- \( \lambda_{\text{medium}} = \frac{\lambda}{n} \)

- Isotropic materials vs. anisotropic materials

- Frequency dependence of \( \varepsilon_r \) — Example 1.2.1
### EXAMPLE 1.2.1 Relative permittivity and refractive index

Relative permittivity \( \varepsilon \), or the dielectric constant of materials, in general, depends on the frequency of the electromagnetic wave. The relationship \( n = \sqrt{\varepsilon} \) between the refractive index \( n \) and \( \varepsilon \) must be applied at the same frequency for both \( n \) and \( \varepsilon \). The relative permittivity for many materials can be vastly different at high and low frequencies because different polarization mechanisms operate at these frequencies. At low frequencies all polarization mechanisms present can contribute to \( \varepsilon \), whereas at optical frequencies only the electronic polarization can respond to the oscillating field. Table 1.1 lists the relative permittivity \( \varepsilon_r(\text{LF}) \) at low frequencies (e.g., 0 Hz or 1 kHz as would be measured for example using a capacitance bridge in the laboratory) for various materials. It then compares \( \sqrt{\varepsilon_r(\text{LF})} \) with \( n \).

For silicon and diamond there is an excellent agreement between \( \varepsilon_r(\text{LF}) \) and \( n \). Both are covalent solids in which electronic polarization (electronic bond polarization) is the only polarization mechanism at low and high frequencies. Electronic polarization involves the displacement of light electrons with respect to heavy positive ions of the crystal. This process can readily respond to the field oscillations up to optical or even ultraviolet frequencies.

<table>
<thead>
<tr>
<th>Material</th>
<th>( \varepsilon_r(\text{LF}) )</th>
<th>( \sqrt{\varepsilon_r(\text{LF})} )</th>
<th>( n ) (optical)</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Si</td>
<td>11.9</td>
<td>3.44</td>
<td>3.95 (at 2.15 ( \mu )m)</td>
<td>Electronic bond polarization up to optical frequencies</td>
</tr>
<tr>
<td>Diamond</td>
<td>5.7</td>
<td>2.39</td>
<td>2.41 (at 590 nm)</td>
<td>Electronic bond polarization up to UV light</td>
</tr>
<tr>
<td>GaAs</td>
<td>13.1</td>
<td>3.62</td>
<td>3.30 (at 3 ( \mu )m)</td>
<td>Ionic polarization contributes to ( \varepsilon_r(\text{LF}) )</td>
</tr>
<tr>
<td>SiO(_2)</td>
<td>3.84</td>
<td>2.06</td>
<td>1.46 (at 600 nm)</td>
<td>Ionic polarization contributes to ( \varepsilon_r(\text{LF}) ), which is large</td>
</tr>
<tr>
<td>Water</td>
<td>80</td>
<td>8.9</td>
<td>1.33 (at 600 nm)</td>
<td>Dipolar polarization contributes to ( \varepsilon_r(\text{LF}) )</td>
</tr>
</tbody>
</table>

For GaAs and SiO\(_2\) \( \sqrt{\varepsilon_r(\text{LF})} \), the values are larger than \( n \) because at low frequencies both of these solids possess a degree of ionic polarization. The bonding is not totally covalent and there is a degree of ionic bonding that contributes to polarization at frequencies below far-infrared wavelengths.

In the case of water, the \( \varepsilon_r(\text{LF}) \) is dominated by orientational or dipolar polarization, which is far too sluggish to respond to high frequency oscillations of the field at optical frequencies.

It is instructive to consider what factors affect \( n \). The simplest (and approximate) expression for relative permittivity is

\[
\varepsilon_r \approx 1 + N\alpha/\varepsilon_0
\]

in which \( N \) is the number of molecules per unit volume and \( \alpha \) is the polarizability per molecule. Both atomic concentration, or density, and polarizability therefore increase \( n \). For example, glasses of given type but with greater density tend to have higher \( n \).
EXAMPLE 1.3.1  Group velocity

Consider two sinusoidal waves that are close in frequency, that is, waves of frequencies $\omega - \delta \omega$ and $\omega + \delta \omega$ as in Figure 1.6. Their wave vectors will be $k - \delta k$ and $k + \delta k$. The resultant wave will be

$$E_z(z, t) = E_o \cos[(\omega - \delta \omega)t - (k - \delta k)z] + E_o \cos[(\omega + \delta \omega)t - (k + \delta k)z]$$

By using the trigonometric identity $\cos A + \cos B = 2 \cos\left[\frac{1}{2}(A - B)\right] \cos\left[\frac{1}{2}(A + B)\right]$ we arrive at

$$E_z(z, t) = 2E_o \cos[(\delta \omega)t - (\delta k)z] \cos[\omega t - kr]$$

As depicted in Figure 1.6, this represents a sinusoidal wave of frequency $\omega$, which is amplitude modulated by a very slowly varying sinusoidal of frequency $\delta \omega$. The system of waves, that is, the modulation, travels along $z$ at a speed determined by the modulating term, $\cos[(\delta \omega)t - (\delta k)z]$. The maximum in the field occurs when $[(\delta \omega)t - (\delta k)z] = 2m \pi = \text{constant}$ ($m$ is an integer), which travels with a velocity

$$\frac{dz}{dt} = \frac{\delta \omega}{\delta k} \quad \text{or} \quad v_g = \frac{\delta \omega}{\delta k}$$

This is the group velocity of the waves, as stated in Eq. (1), since it determines the speed of propagation of the maximum electric field along $z$.

Group velocity and group index

- Group velocity $v_g = \frac{\delta \omega}{\delta k}$

$$\omega = v'k = \frac{c}{n(\lambda)}\frac{\delta \omega}{\delta \lambda} \quad \Rightarrow \quad v_g = \frac{\delta \omega}{\delta k} = \frac{c}{n - \lambda \frac{dn}{d\lambda}}$$

- Group index $N_g = \frac{c}{v_g} = n - \lambda \frac{dn}{d\lambda}$

Two slightly different wavelength waves travelling in the same direction result in a wave packet that has an amplitude variation which travels at the group velocity.
Refractive index $n$ and the group index $N_g$ of pure SiO$_2$ (silica) glass as a function of wavelength.

**EXAMPLE 1.3.2 Group and phase velocities**

Consider a light wave traveling in a pure SiO$_2$ (silica) glass medium. If the wavelength of light is 1 µm and the refractive index at this wavelength is 1.450, what is the phase velocity, group index ($N_g$) and group velocity ($v_g$)?

**Solution** The phase velocity is given by

$$v = c/n = (3 \times 10^8 \text{ m s}^{-1})/(1.450)$$

$$= 2.069 \times 10^8 \text{ m s}^{-1}.$$ 

From Figure 1.7, at $\lambda = 1 \mu\text{m}$, $N_g = 1.460$, so that

$$v_g = c/N_g = (3 \times 10^8 \text{ ms}^{-1})/(1.460)$$

$$= 2.055 \times 10^8 \text{ m s}^{-1}.$$ 

The group velocity is about ~0.7% slower than the phase velocity.
Magnetic field, irradiance and Poynting vector

\[ E_x = \nu B_y = \frac{c}{n} B_y \]

\[ (\nu = \sqrt{\varepsilon_0 \varepsilon_r \mu_0}, \quad n = \sqrt{\varepsilon_r}) \]

- Energy densities in an EM wave
  \[ \frac{1}{2} \varepsilon_0 \varepsilon_r E_x^2 = \frac{1}{2\mu_0} B_y^2 \]

- Energy flow per unit time per unit area
  \[ S = \frac{(A \Delta t) (\varepsilon_0 \varepsilon_r E_x^2)}{A \Delta t} = \nu \varepsilon_0 \varepsilon_r E_x^2 = \nu^2 \varepsilon_0 \varepsilon_r E_x B_y \]

- Poynting vector
  \[ \vec{S} = \vec{E} \times \vec{H} = \nu^2 \varepsilon_0 \varepsilon_r \vec{E} \times \vec{B} \]

  \[ |\vec{S}| = \text{irradiance} \]

- Average irradiance (intensity)
  \[ I = \langle S \rangle = \frac{1}{2} \nu \varepsilon_0 \varepsilon_r \frac{\varepsilon_0^2}{n} \]

A plane EM wave travelling along \( \vec{k} \) crosses an area \( A \) at right angles to the direction of propagation. In time \( \Delta t \), the energy in the cylindrical volume \( A \nu \Delta t \) (shown dashed) flows through \( A \).

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EXAMPLE 1.4.1 Electric and magnetic fields in light

The intensity (irradiance) of the red laser beam from a He-Ne laser at a certain location has been measured to be about 1 mW cm$^{-2}$. What are the magnitudes of the electric and magnetic fields? What are the magnitudes if this beam is in a glass medium with a refractive index $n = 1.45$?

**Solution** Using Eq. (6) for the average irradiance, the field in air is

$$E_o = \sqrt{\frac{2I}{\varepsilon_0 n}} = \sqrt{\frac{2(1 \times 10^{-3} \times 10^4 \text{ W m}^{-2})}{(3 \times 10^8 \text{ m s}^{-1})(8.85 \times 10^{-12} \text{ F m}^{-1})}} \quad (1)$$

so that

$$E_o = 87 \text{ V m}^{-1} \quad \text{or} \quad 0.87 \text{ V cm}^{-1}.$$

The corresponding magnetic field is

$$B_o = \frac{E_o}{c} = \frac{(0.87 \text{ V m}^{-1})}{(3 \times 10^8 \text{ m s}^{-1})} = 0.29 \mu\text{T}.$$

If this 1 mW cm$^{-2}$ beam were in a glass medium of $n = 1.45$ and still had the same intensity, then

$$E_o(\text{medium}) = \sqrt{\frac{2I}{\varepsilon_0 n}} = \sqrt{\frac{2(1 \times 10^{-3} \times 10^4 \text{ W m}^{-2})}{(3 \times 10^8 \text{ m s}^{-1})(8.85 \times 10^{-12} \text{ F m}^{-1})(1.45)}} \quad (1.45)$$

or

$$E_o(\text{medium}) = 72 \text{ V m}^{-1}$$

and

$$B_o(\text{medium}) = \frac{nE_o(\text{medium})}{c} = \frac{(1.45)(72 \text{ V m}^{-1})}{(3 \times 10^8 \text{ m s}^{-1})} = 0.35 \mu\text{T}.$$
Snell’s Law

A light wave travelling in a medium with a greater refractive index \( n_1 > n_2 \) suffers reflection and refraction at the boundary.

The time it takes to go from \( B \) to \( B' = \frac{BB'}{c/n_1} = t_1 \) \( \Rightarrow \theta_i = \theta_r \)

\[
\begin{align*}
t_i &= \frac{AB' \sin \theta_i}{c/n_1} \\
t_2 &= \frac{AB' \sin \theta_r}{c/n_1} \\
t_3 &= \frac{n_2 \sin \theta_t}{c/n_2} = n_1 \sin \theta_i
\end{align*}
\]
Total Internal Reflection (TIR)

Light wave travelling in a more dense medium strikes a less dense medium. Depending on the incidence angle with respect to $\theta_c$, which is determined by the ratio of the refractive indices, the wave may be transmitted (refracted) or reflected. (a) $\theta_i < \theta_c$ (b) $\theta_i = \theta_c$ (c) $\theta_i > \theta_c$ and total internal reflection (TIR).

When $n_1 > n_2 \Rightarrow \theta_t > \theta_i$

Critical angle: when $\theta_t = 90^\circ$, $\theta_i = \theta_c$

$$n_1 \sin \theta_c = n_2 \sin 90^\circ \Rightarrow \theta_c = \sin^{-1} \left( \frac{n_2}{n_1} \right)$$

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Transmission and Reflection

(a) $\theta_i < \theta_c$ then some of the wave is transmitted into the less dense medium. Some of the wave is reflected.

(b) $\theta_i > \theta_c$ then the incident wave suffers total internal reflection. However, there is an evanescent wave at the surface of the medium.

Light wave travelling in a more dense medium strikes a less dense medium. The plane of incidence is the plane of the paper and is perpendicular to the flat interface between the two media. The electric field is normal to the direction of propagation. It can be resolved into perpendicular (\(\perp\)) and parallel (\(//\)) components.

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Fresnel Equations

Transverse electric field (TE) wave (s wave) \( E_\perp \)

Transverse magnetic field (TM) wave (p wave) \( E_{\parallel} \)

Incidence wave \( E_i = E_{i0} e^{i(\omega t - \vec{k}_i \cdot \vec{r})} \)

Reflected wave \( E_r = E_{r0} e^{i(\omega t - \vec{k}_r \cdot \vec{r})} \)

Transmitted wave \( E_t = E_{t0} e^{i(\omega t - \vec{k}_t \cdot \vec{r})} \)

Reflection coefficient \( \gamma = \frac{E_{r0}}{E_{i0}} \)

Transmission coefficient \( t = \frac{E_{t0}}{E_{i0}} \)

Boundary Conditions:

\( E_{\text{tangential}} (1) = E_{\text{tangential}} (2) \)

\( B_{\text{tangential}} (1) = B_{\text{tangential}} (2) \)
Transmission and Reflection Coefficients

\[ S \text{ wave:} \]
\[
\begin{align*}
\tau_\perp &= \frac{E_{r,\perp}}{E_{i,\perp}} = \frac{n_1 \cos \theta_i - n_2 \cos \theta_t}{n_1 \cos \theta_i + n_2 \cos \theta_t} \\
t_\perp &= \frac{E_{t,\perp}}{E_{i,\perp}} = \frac{2n_1 \cos \theta_i}{n_1 \cos \theta_i + n_2 \cos \theta_t}
\end{align*}
\]
\[ \tau_\perp + 1 = t_\perp \]

\[ p \text{ wave:} \]
\[
\begin{align*}
\tau_\parallel &= \frac{E_{r,\parallel}}{E_{i,\parallel}} = \frac{n_1 \cos \theta_i - n_2 \cos \theta_t}{n_1 \cos \theta_i + n_2 \cos \theta_t} \\
t_\parallel &= \frac{E_{t,\parallel}}{E_{i,\parallel}} = \frac{2n_1 \cos \theta_i}{n_1 \cos \theta_i + n_2 \cos \theta_t}
\end{align*}
\]
\[ \tau_\parallel + \frac{n_2}{n_1} t_\parallel = 1 \]

Brewster's angle (polarization angle): \[ \tan \theta_p = \frac{n_2}{n_1} \]

TIR: evanescent wave
\[
E_{t,\perp}(y,z,t) = e^{-\alpha_2 y} e^{i(\omega t - k_2 z)}
\]
\[
\alpha_2 = \frac{2\pi n_2}{\lambda} \left[ \left( \frac{n_1}{n_0} \right)^2 \sin^2 \theta_i - 1 \right]^{1/2}
\]
Internal reflection: (a) Magnitude of the reflection coefficients $r_{//}$ and $r_{\perp}$ vs. angle of incidence $\theta_i$ for $n_1 = 1.44$ and $n_2 = 1.00$. The critical angle is $44^\circ$. (b) The corresponding phase changes $\phi_{//}$ and $\phi_{\perp}$ vs. incidence angle.
The reflection coefficients $r_\parallel$ and $r_\perp$ vs. angle of incidence $\theta_i$ for $n_1 = 1.00$ and $n_2 = 1.44$. 

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EXAMPLE 1.6.1  Evanescent wave

Total internal reflection (TIR) of light from a boundary between a more dense medium $n_1$ and a less dense medium $n_2$ is accompanied by an evanescent wave propagating in medium 2 near the boundary. Find the functional form of this wave and discuss how its magnitude varies with the distance into medium 2.

Solution  The transmitted wave has the general form

$$E_{t,\perp} = t_\perp E_{i0,\perp} \exp(j(\omega t - \mathbf{k}_t \cdot \mathbf{r}))$$

in which $t_\perp$ is the transmission coefficient. The dot product, examining Figure 1.3, is

$$\mathbf{k}_t \cdot \mathbf{r} = yk_t \cos \theta_t + zk_t \sin \theta_t.$$

However, from Snell's law, when $\theta_t > \theta_c$, $\sin \theta_t = (n_1/n_2) \sin \theta_i > 1$ and $\cos \theta_t = \sqrt{1 - \sin^2 \theta_t}$

$$= \pm jA_2$$

is a purely imaginary number. Thus, taking $\cos \theta_t = -jA_2$

$$E_{t,\perp} = t_\perp E_{i0,\perp} \exp(j(\omega t - zk_t \sin \theta_t + jyk_t A_2))$$

$$= t_\perp E_{i0,\perp} \exp(-yk_t A_2) \exp(j(\omega t - zk_t \sin \theta_t))$$

which has an amplitude that decays along $y$ as $\exp(-\alpha_2 y)$ where $\alpha_2 = k_t A_2$. Note that $+jA_2$ is ignored because it implies a light wave in medium 2 whose amplitude and hence intensity grows.

Consider the traveling wave part $\exp(j(\omega t - zk_t \sin \theta_t))$. Here, $k_t \sin \theta_t = k_t \sin \theta_t$ (by virtue of Snell's law). But $k_t \sin \theta_t = k_t z$, which is the wavevector along $z$, that is, along the boundary. Thus the evanescent wave propagates along $z$ at the same speed as the incident and reflected waves along $z$.

Furthermore, for TIR we need $\sin \theta_i > n_2/n_1$. This means that the transmission coefficient,

$$t_\perp = \frac{n_i \cos \theta_i}{\cos \theta_t + \left[\left(\frac{n_2}{n_1}\right)^2 - \sin^2 \theta_t\right]^{1/2}} = t_{i0} \exp(j\psi_\perp)$$

must be a complex number as indicated by $t_{i0} \exp(j\psi_\perp)$ in which $t_{i0}$ is a real number and $\psi_\perp$ is a phase change. Note that $t_\perp$ does not, however, change the general behavior of propagation along $z$ and the penetration along $y$. 
Intensity, Reflectance, and Transmittance

\[ I = \frac{1}{2} \, \nu \varepsilon_r \varepsilon_0 \, E_0^2 = \frac{1}{2} \, \varepsilon_0 \, n \varepsilon_0^2 \]
\[ \nu = \frac{c}{n}, \quad \varepsilon_r = n^2 \]

Reflectance
\[ R_\perp = \frac{|E_{\text{io} \perp}|^2}{|E_{\text{io} \perp}|^2} = |r_\perp|^2 \]
\[ R_\parallel = \frac{|E_{\text{io} \parallel}|^2}{|E_{\text{io} \parallel}|^2} = |r_\parallel|^2 \]

At normal incidence
\[ r_\parallel = r_\perp = \frac{n_1 - n_2}{n_1 + n_2} \]
\[ R = R_\parallel = R_\perp = \left( \frac{n_1 - n_2}{n_1 + n_2} \right)^2 \]

Transmittance
\[ T_\perp = \frac{n_2 |E_{\text{to} \perp}|^2 \cos \theta_i}{n_1 |E_{\text{io} \perp}|^2 \cos \theta_i} = \frac{n_2 \cos \theta_i}{n_1 \cos \theta_i} |t_\perp|^2 \]
\[ T_\parallel = \frac{n_2 \cos \theta_i |E_{\text{to} \parallel}|^2}{n_1 \cos \theta_i |E_{\text{io} \parallel}|^2} = \frac{n_2 \cos \theta_i}{n_1 \cos \theta_i} |t_\parallel|^2 \]

Energy Conservation:
\[ R + T = 1 \]
Total Internal Reflection (TIR)

Critical Angle:

\[ \sin \theta_c = \frac{n_2}{n_1} \]

When \( \theta_i > \theta_c \)

\[ \sin \theta_t = \frac{n_1}{n_2} \sin \theta_i \]

\[ \cos \theta_t = -j \sqrt{\sin^2 \theta_i - 1} \]

Phase change:

\[ \tan(\phi_\perp / 2) = \frac{\sqrt{\sin^2 \theta_i - \sin^2 \theta_c}}{\cos \theta_i} \]
EXAMPLE 1.6.2  Reflection of light from a less dense medium
(internal reflection)

A ray of light that is traveling in a glass medium of refractive index \( n_1 = 1.450 \) becomes incident on a less dense glass medium of refractive index \( n_2 = 1.430 \). Suppose that the free space wavelength (\( \lambda \)) of the light ray is 1 \( \mu \text{m} \).

a. What should the minimum incidence angle for TIR be?

b. What is the phase change in the reflected wave when \( \theta_i = 85^\circ \) and when \( \theta_i = 90^\circ \)?

c. What is the penetration depth of the evanescent wave into medium 2 when \( \theta_i = 85^\circ \) and when \( \theta_i = 90^\circ \)?

Solution

a. The critical angle \( \theta_c \) for TIR is given by \( \sin \theta_c = n_2/n_1 = 1.430/1.450 \) so that \( \theta_c = 80.47^\circ \).

b. Since the incidence angle \( \theta_i > \theta_c \), there is a phase shift in the reflected wave. The phase change in \( E_{r,\perp} \) is given by \( \phi_\perp \). With \( n_1 = 1.450, n_2 = 1.430 \), and \( \theta_i = 85^\circ \),

\[
\tan(\frac{1}{2} \phi_\perp) = \frac{[\sin^2 \theta_i - n^2]^\frac{1}{2}}{\cos \theta_i} = \frac{[\sin^2 (85^\circ) - (1.420/1.450)^2]^\frac{1}{2}}{\cos (85^\circ)} = 1.61447 = \tan(\frac{1}{2} (116.45^\circ))
\]

so that the phase change is 116.45\(^\circ\). For the \( E_{r,\parallel} \) component, the phase change is

\[
\tan\left(\frac{1}{2} \phi_{\parallel} + \frac{1}{2} \pi \right) = \frac{[\sin^2 \theta_i - n^2]^\frac{1}{2}}{n^2 \cos \theta_i} = \frac{1}{n^2} \tan(\frac{1}{2} \phi_\perp)
\]

so that

\[
\tan\left(\frac{1}{2} \phi_{\parallel} + \frac{1}{2} \pi \right) = (n_1/n_2)^2 \tan(\phi_\parallel/2) = (1.450/1.430)^2 \tan(\frac{1}{2} 116.45^\circ)
\]

which gives \( \phi_{\parallel} = -62.1^\circ \). (Note: If we were to invert the reflected field, this phase change would be +117.86\(^\circ\))

We can repeat the calculation with \( \theta_i = 90^\circ \) to find \( \phi_\perp = 180^\circ \) and \( \phi_{\parallel} = 0^\circ \).

Note that as long as \( \theta_i > \theta_c \), the magnitude of the reflection coefficients are unity. Only the phase changes.
c. The amplitude of the evanescent wave as it penetrates into medium 2 is

\[ E_{\text{ev},1}(y, t) \sim E_{\text{ev},1} \exp(-\alpha_2 y) \]

We ignore the \( z \)-dependence, \( \exp(\imath \omega t - k_z z) \), as this only gives a propagating property along \( z \). The field strength drops to \( e^{-1} \) when \( y = 1/\alpha_2 = \delta \), which is called the penetration depth. The attenuation constant \( \alpha_2 \) is

\[ \alpha_2 = \frac{2 \pi n_2}{\lambda} \left[ \left( \frac{n_1}{n_2} \right)^2 \sin^2 \theta_i - 1 \right]^{1/2} \]

\[ \alpha_2 = \frac{2 \pi (1.430)}{(1.0 \times 10^{-6} \text{ m})} \left[ \left( \frac{1.450}{1.430} \right)^2 \sin^2 (85^\circ) - 1 \right]^{1/2} = 1.28 \times 10^6 \text{ m}^{-1}. \]

so the penetration depth is \( \delta = 1/\alpha_2 = 1/(1.28 \times 10^6) = 7.8 \times 10^{-7} \text{ m}, \) or 0.78 \( \mu \text{m}. \)

For 90\(^\circ\), repeating the calculation we find \( \alpha_2 = 1.5 \times 10^6 \text{ m}^{-1}, \) so that \( \delta = 1/\alpha_2 = 0.66 \mu \text{m}. \)

We see that the penetration is greater for smaller incidence angles. This will be an important consideration later in analyzing light propagation in optical fibers.
EXAMPLE 1.6.3  Reflection at normal incidence. Internal and external reflection

Consider the reflection of light at normal incidence on a boundary between a glass medium of refractive index 1.5 and air of refractive index 1.

a. If light is traveling from air to glass, what is the reflection coefficient and the intensity of the reflected light with respect to that of the incident light?

b. If light is traveling from glass to air, what is the reflection coefficient and the intensity of the reflected light with respect to that of the incident light?

c. What is the polarization angle in the external reflection in a above? How would you make a polaroid device that polarizes light based on the polarization angle?

Solution

a. The light travels in air and becomes partially reflected at the surface of the glass that corresponds to external reflection. Thus $n_1 = 1$ and $n_2 = 1.5$. Then,

$$r_\parallel = r_\perp = \frac{n_1 - n_2}{n_1 + n_2} = \frac{1 - 1.5}{1 + 1.5} = -0.2$$

This is negative, which means that there is a $180^\circ$ phase shift. The reflectance ($R$), which gives the fractional reflected power, is

$$R = r_\parallel^2 = 0.04 \quad \text{or} \quad 4\%.$$

b. The light travels in glass and becomes partially reflected at the glass-air interface that corresponds to internal reflection. Thus $n_1 = 1.5$ and $n_2 = 1$. Then,

$$r_\parallel = r_\perp = \frac{n_1 - n_2}{n_1 + n_2} = \frac{1.5 - 1}{1.5 + 1} = 0.2$$

There is no phase shift. The reflectance is again 0.04 or 4%. In both cases (a) and (b), the amount of reflected light is the same.
c. Light is traveling in air and is incident on the glass surface at the polarization angle. Here \( n_1 = 1, n_2 = 1.5 \) and \( \tan \theta_p = (n_2/n_1) = 1.5 \) so that \( \theta_p = 56.3^\circ \). (We can use Fresnel's equations to readily find the reflected and transmitted amplitudes.)

If we were to reflect light from a glass plate, keeping the angle of incidence at 56.3°, then the reflected light will be polarized with an electric field component perpendicular to the plane of incidence. The transmitted light will have the field greater in the plane of incidence, that is, it will be partially polarized. By using a stack of glass plates one can increase the polarization of the transmitted light. (This type of pile-of-plates polarizer was invented by Dominique F.J. Arago in 1812.)
Antireflection coatings on solar cells

When light is incident on the surface of a semiconductor, it becomes partially reflected. Partial reflection is an important consideration in solar cells where transmitted light energy into the semiconductor device is converted to electrical energy. The refractive index of Si is about 3.5 at wavelengths around 700–800 nm. Thus the reflectance with \( n_1(\text{air}) = 1 \) and \( n_2(\text{Si}) \approx 3.5 \) is

\[
\begin{align*}
n & = \left( \frac{n_1^2 - n_2^2}{n_1^2 + n_2^2} \right)^2 \approx \left( \frac{1 - 3.5}{1 + 3.5} \right)^2 \approx 0.099
\end{align*}
\]

This means that 31% of light is reflected and not available for conversion to electrical energy; a considerable reduction in the efficiency of the solar cell.

However, we can coat the surface of the semiconductor device with a thin layer of a dielectric material, such as SiO\(_2\) (silicon nitride), that has an intermediate refractive index. Figure 1.14 illustrates how the thin dielectric coating reduces the reflected light intensity. In this case \( n_1(\text{air}) = 1, n_2(\text{coating}) \approx 1.9, \) and \( n_3(\text{Si}) = 3.5. \) Light is first incident on the air/coating surface and some of it becomes reflected and this reflected wave is shown as \( A \) in Figure 1.14. Wave \( A \) has experienced a 180° phase change on reflection as this is an external reflection. The wave that enters and travels in the coating then becomes reflected at the coating/silicon interface. This wave, which is shown as \( B \) in Figure 1.14, also suffers a 180° phase change since \( n_3 \geq n_2. \) When wave \( B \) reaches \( A, \) it has suffered a total delay of traversing the thickness \( d \) of the coating twice. The phase difference is equivalent to \( k_c(2d) \) in which \( k_c = 2\pi/\lambda_c \) is the wavevector in the coating and is given by \( 2\pi/\lambda_c, \) in which \( \lambda_c \) is the wavelength in the coating. Since \( \lambda_c = \lambda/n_2, \) where \( \lambda \) is the free-space wavelength, the phase difference \( \Delta\phi \) between \( A \) and \( B \) is \( (2\pi n_2/\lambda)(2d). \) To reduce the reflected light, \( A \) and \( B \) must interfere destructively and this requires the phase difference to be \( \pi \) or odd-multiples of \( \pi, m\pi \) in which \( m = 1, 3, 5, \ldots \) is an odd-integer. Thus

\[
\left( \frac{2\pi n_2}{\lambda} \right) 2d = m\pi \quad \text{or} \quad \lambda = m\left( \frac{4n_2}{d} \right)
\]

Thus, the thickness of the coating must be multiples of the quarter wavelength in the coating and depends on the wavelength.

To obtain a good degree of destructive interference between waves \( A \) and \( B, \) the two amplitudes must be comparable. It turns out that we need \( n_2 = \sqrt{(n_1 n_3)}. \) When \( n_2 = \sqrt{(3.5 \times 1.9)} \) then the reflection coefficient between the air and coating is equal to that between the coating and the semiconductor. In this case we would need \( \sqrt{(3.5)} \) or 1.87. Thus, SiO\(_2\) is a good choice as an antireflection coating material on Si solar cells.

Taking the wavelength to be 700 nm, \( d = (700 \text{ nm})/\left[4(1.9)\right] = 92.1 \text{ nm} \) or odd-multiples of \( d. \)
**EXAMPLE 1.6.5  Dielectric mirrors**

A **dielectric mirror** consists of a stack of dielectric layers of alternating refractive indices as schematically illustrated in Figure 1.15, in which \( n_1 \) is smaller than \( n_2 \). The thickness of each layer is a quarter of wavelength or \( \lambda_{\text{layer}}/4 \) in which \( \lambda_{\text{layer}} \) is the wavelength of light in that layer, or \( \lambda_o/n \) in which \( \lambda_o \) is the free space wavelength at which the mirror is required to reflect the incident light and \( n \) is the refractive index of the layer. Reflected waves from the interfaces interfere constructively and give rise to a substantial reflected light. If there are sufficient number of layers, the reflectance can approach unity at the wavelength \( \lambda_o \). The figure also shows schematically a typical reflectance vs. wavelength behavior of a dielectric mirror with many layers.

The reflection coefficient \( r_{12} \) for light in layer 1 being reflected at the 1-2 boundary is \( r_{12} = (n_1 - n_2)/(n_1 + n_2) \) and is a negative number indicating a \( \pi \) phase change. The reflection coefficient for light in layer 2 being reflected at the 2-1 boundary is \( r_{21} = (n_2 - n_1)/(n_2 + n_1) \), which is \(-r_{12}\) (positive) indicating no phase change. Thus the reflection coefficient alternates in sign through the mirror. Consider two arbitrary waves, \( A \) and \( B \), which are reflected at two consecutive interfaces. The two waves are therefore already out of phase by \( \pi \) due to reflections at the different boundaries. Further, wave \( B \) travels an additional distance that is twice \( (\lambda_{\text{layer}}/2) \) before reaching wave \( A \) and therefore experiences a phase change equivalent to \( 2(\lambda_{\text{layer}}/4) \) or \( \lambda_{\text{layer}}/2 \), that is \( \pi \). The phase difference between \( A \) and \( B \) is then \( \pi + \pi \) or \( 2\pi \). Thus waves \( A \) and \( B \) are in phase and interfere constructively. We can similarly show that waves \( B \) and \( C \) also interfere constructively and so on, so that all reflected waves from the consecutive boundaries interfere constructively. After several layers (depending on \( n_1 \) and \( n_2 \)) the transmitted intensity will be very small and the reflected light intensity will be close to unity. Dielectric mirrors are widely used in modern vertical cavity surface emitting semiconductor lasers.

![Schematic illustration of the principle of the dielectric mirror with many low and high refractive index layers and its reflectance.](image)