Lecture 8

Dielectric Waveguides and Optical Fibers

- Slab Waveguide, Modes, V-Number
- Modal, Material, and Waveguide Dispersions
- Step-Index Fiber, Multimode and Single Mode Fibers
- Numerical Aperture, Coupling Loss
- Bit-Rate, dispersion and optical bandwidth
- Graded-index fibers
- Absorption and Scattering
- Fiber Manufacture
Waveguide Condition & Waveguide Modes

- Waveguide Condition:
  \[
  \frac{2\pi n_1 (2a)}{\lambda} \cos \theta_m - \varphi_m = m\pi
  \]
  a is the radius for optical fibers!

- Fresnel’s Equation:
  \[
  \tan \left( \frac{1}{2} \varphi_m \right) = \frac{\sin^2 \theta_m - \left( \frac{n_2}{n_1} \right)^2}{\cos \theta_m} \]
  solutions \( m, \theta_m \)
Waveguide Modes & Field Profile

<table>
<thead>
<tr>
<th>$m$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
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</thead>
<tbody>
<tr>
<td>$\theta_m$</td>
<td>89.2°</td>
<td>88.3°</td>
<td>87.5°</td>
<td>86.7°</td>
<td>85.9°</td>
<td>85.0°</td>
<td>84.2°</td>
<td>83.4°</td>
<td>82.6°</td>
<td>81.9°</td>
</tr>
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</table>

$m = 0$
Fundamental mode

Critical angle $\theta_c = \arcsin(n_2/n_1) = 81.77°$

Each mode $m$ is a possible EM wave that can propagate in the fiber

$solutions \ m = 0, 1, 2\ldots$
Propagation Constants for Modes

**Propagation constant along the guide**

\[ \beta_m = k_1 \sin \theta_m = \left( \frac{2\pi n_1}{\lambda} \right) \sin \theta_m \]

**Transverse Propagation constant**

\[ \kappa_m = k_1 \cos \theta_m = \left( \frac{2\pi n_1}{\lambda} \right) \cos \theta_m \]

The upward and downward traveling waves interfere to set up a standing electric field pattern across the guide.

\[ E(y,z,t) = E_m(y) \cos(\omega t - \beta_m z) \]

Field pattern along y

Traveling wave along z
Waveguide Modes

Modes of a Vibrating String

- Fundamental (1st harmonic)
- 2nd harmonic
- 3rd harmonic

Diagram showing waveguide modes with a core and cladding layers.
Single and Multimode Waveguides

\[
\left[ \frac{2\pi n_1(2a)}{\lambda} \right] \cos \theta_m - \phi_m = m\pi \quad \& \quad \theta_m \approx \theta_c \quad \Rightarrow \quad \sin \theta_c = \frac{n_2}{n_1}
\]

Waveguide condition

In the Limit

\[
N = \text{Int} \left( \frac{2V}{\pi} \right) + 1
\]

Number of Modes

V-number or normalized frequency

One mode when \( V < \pi/2 \)  
Multimode when \( V > \pi/2 \)

\[
\lambda_c = 4a \left( n_1^2 - n_2^2 \right)^{1/2}
\]

Cut-Off Wavelength !!
Number of Modes

\[ \tan(ak_1 \cos \theta_m - m \pi/2) \]

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Using Eq. (2.1.7), estimate the number of modes that can be supported in a planar dielectric waveguide that is 100 \( \mu \text{m} \) wide, and has \( n_1 = 1.490 \) and \( n_2 = 1.470 \) at the free-space source wavelength \( (\lambda) \), which is 1 \( \mu \text{m} \). Compare your estimate with the formula

\[
M = \text{Int}\left(\frac{2V}{\pi}\right) + 1
\]  

(2.1.11)

in which Int\( (x) \) is the integer function; it drops the decimal fraction of \( x \).

The phase change \( \phi \) on TIR cannot be more than \( \pi \) so \( \phi/\pi \) is less than 1. For a multimode guide \( (V \gg 1) \), we can write Eq. (2.1.7) as

\[
m \leq \frac{2V - \phi}{\pi} \approx \frac{2V}{\pi}
\]

We can calculate \( V \) since we are given \( a = 50 \ \mu\text{m}, \ n_1 = 1.490, \ n_2 = 1.460, \) and \( \lambda = 1 \times 10^{-6} \text{ m} \) or 1 \( \mu \text{m} \),

\[
V = (2\pi a/\lambda)(n_1^2 - n_2^2)^{1/2} = 76.44
\]

Then, \( m \leq 2(76.44)/\pi = 48.7 \), or \( m \leq 48 \). There are about 49 modes, because we must also include \( m = 0 \) as a mode. Using Eq. (2.1.11),

\[
M = \text{Int}\left[\frac{2(76.44)}{\pi}\right] + 1 = 49
\]
EM Waves in Free Space

Plane Wave:

$$E_x = E_0 \cos(\omega t - k z + \phi_0)$$

- $\omega$ = angular frequency
- $t$ = time
- $k$ = propagation constant (wave number)
  $$k = \frac{2\pi}{\lambda}, \quad \lambda = \text{wavelength}$$
- $z$ = position
- $\phi_0$ = phase constant

Frequency

$\beta_m$

Slope = $c/n_1$

Slope = $c/n_2$

$n_2 < n_1$
Waveguide Dispersion Diagram (Wavelength Dependence)

Waveguide condition

\[ \frac{2\pi n_1 (2a)}{\lambda} \cos \theta_m - \varphi_m = m\pi \]

Propagation constant

\[ \beta_m = k_1 \sin \theta_m = \left( \frac{2\pi n_1}{\lambda} \right) \sin \theta_m \]

\[ \frac{c}{2\pi n_1 \sin \theta_m} \beta_m = \omega \]

The slope of \( \omega \) vs. \( \beta \) is the group velocity \( v_g \)
Refractive Index & Group Index

Group Velocity

\[ v_g(\text{medium}) = \frac{d\omega}{dk} = \frac{c}{n - \lambda \frac{dn}{d\lambda}} \]

\[ N_g = n - \lambda \frac{dn}{d\lambda} \]
Intermodal Dispersion

Group Velocity (Mode)

\[ V_g = \frac{d\omega}{d\beta_m} \]

Group velocity vs. frequency or wavelength behavior is not obvious. For the first few modes, a higher mode can travel faster than the fundamental.

The group velocity \( V_g \) vs. \( \omega \) for a planar dielectric guide with a core thickness \((2a) = 20 \text{ \mu m}\), \( n_1 = 1.455 \), \( n_2 = 1.440 \). \( \text{TE}_0 \), \( \text{TE}_1 \) and \( \text{TE}_4 \).
Dispersion Near Cut-Off (TE$_0$ and TE$_1$)

Group velocity vs. frequency or wavelength behavior is not obvious. For the first few modes, a higher mode can travel faster than the fundamental.

\[ \omega_{\text{cutoff}} = 2.3 \times 10^{14} \text{ rad/s} \]

- \( v_{g_{\text{max}}} \approx c/n_2 \)
- \( v_{g_{\text{min}}} \approx c/n_1 \)

Spread in arrival times:

\[ \Delta \tau = \frac{L}{v_{g_{\text{min}}}} - \frac{L}{v_{g_{\text{max}}}} \]

Dispersion:

\[ \frac{\Delta \tau}{L} \approx \frac{n_1 - n_2}{c} \]
Dispersion Near Cut-Off (TE₀ and TE₁)

Group velocity vs. frequency or wavelength behavior is not obvious. For the first few modes, a higher mode can travel faster than the fundamental.

\[ \omega_{\text{cutoff}} = 2.3 \times 10^{14} \quad \text{rad/s} \]

\[ \Delta \tau \approx \frac{n_1 - n_2}{c} \]

\[ L \]

Broadened pulse

Output pulse
The group velocity $v_g$ vs. $w$ for a planar dielectric guide
Core thickness $(2a) = 20$ mm, $n_1 = 1.455$, $n_2 = 1.440$
Dispersion Far From Cutoff

\[ v_{g_{\text{min}}} \approx \frac{c}{n_1} \sin \theta_c = \frac{c}{n_1} \left( \frac{n_2}{n_1} \right) \]

\[ v_{g_{\text{max}}} \approx \frac{c}{n_1} \]
Dispersion Far From Cutoff

\[ \frac{\Delta \tau}{L} = \frac{1}{V_{g_{\min}}} - \frac{1}{V_{g_{max}}} \]

\[ \Delta \tau = \frac{n_1^2}{c n_2} - \frac{n_1}{c} = \frac{1}{c} \left[ \frac{(n_1 - n_2)n_1}{n_2} \right] \approx \frac{(n_1 - n_2)}{c} \]

\[ \frac{\Delta \tau}{L} \approx \frac{n_1 - n_2}{c} \]

(Since \( n_1 \) and \( n_2 \) are only slightly different.)
Intramodal Dispersion

The electric field of TE\(_0\) mode extends more into the cladding as the wavelength increases. As more of the field is carried by the cladding, the group velocity increases.

- Waveguide Dispersion: \[ \frac{2\pi n_1(2a)}{\lambda} \cos \theta_m - \phi_m = m\pi \quad \lambda_1 \neq \lambda_2 \]

- Intramode Dispersion = Waveguide Dispersion + Material Dispersion
Lecture 9

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The central region, the core, has greater refractive index than the outer region, the cladding.

The fiber has cylindrical symmetry. The coordinates $r, \phi, z$ are used to represent any point $P$ in the fiber.

Cladding is normally much thicker than shown.

**Normalized index difference:**

$$\Delta = \frac{n_1 - n_2}{n_1}$$

Need two integers to label a mode: $(l, m)$
**Meridional & Skew Rays**

**Meridional ray** enters the fiber through the fiber axis and hence also crosses the fiber axis on each reflection as it zigzags down the fiber. It travels in a plane that contains the fiber axis.

**Skew ray** enters the fiber off the fiber axis and zigzags down the fiber without crossing the axis.

Meridional Rays $\Rightarrow$ TE\(_{0m}\), TM\(_{0m}\)  

Skew Rays - Hybrid modes $\Rightarrow$ HE\(_{lm}\), EH\(_{lm}\)

$E_z = 0$ or $H_z = 0$  

$E_z \neq 0$ or $H_z \neq 0$
TE & TM Modes

Plane of incidence is the paper.

- Different propagation constants $\phi_\perp \neq \phi_\parallel$
- Different penetration depths
- Electric field is in the direction of propagation!!
- TE & TM waveguide and cutoff condition is accepted same. $n_1 - n_2 \ll 1$

$E_\perp$ is along $x$, so that $E_\perp = E_x$

$B_\perp$ is along $-x$, so that $B_\perp = -B_x$
Modes $\text{LP}_{lm}$ (Linearily Polarized)

Meridional Rays $\Rightarrow \text{TE}_{0m}, \text{TM}_{0m}$

Skew Rays - Hybrid modes $\Rightarrow \text{HE}_{lm}, \text{EH}_{lm}$

$E_z = 0$ or $H_z = 0$

$E_z \neq 0$ or $H_z \neq 0$

Weakly guiding modes in fibers

$\Delta \ll 1$ weakly guiding fibers

$E_{LP} = E_{lm}(r, \phi) \exp(j(\omega t - \beta_{lm}z))$

Field Pattern  Traveling wave

$E_z = 0$ and $H_z = 0$

$E$ and $B$ are $90^\circ$ to each other and $z$
The electric field distribution of the fundamental mode, LP$_{01}$, in the transverse plane to the fiber axis $z$.

The light intensity is greatest at the center of the fiber.
Fundamental Mode (l=0 and m=1)

- $m$ number of maxima along $r$ starting from the core center
- $2l$ number of maxima around a circumference.
- $l$ is the radial mode number. It represents the extent of the helical propagation.
Optical Fiber Parameters (SMF vs MMF)

\[ n = (n_1 + n_2)/2 = \text{average refractive index} \]

\[ \Delta = \text{normalized index difference} \]
\[ \Delta = (n_1 - n_2)/n_1 \approx (n_1^2 - n_2^2)/2 \]

**V-number**

\[ V = \frac{2\pi a}{\lambda} \left( n_1^2 - n_2^2 \right)^{1/2} = \frac{2\pi a}{\lambda} \left( 2n_1 n\Delta \right)^{1/2} \]

\[ V < 2.405 \text{ only 1 mode exists. Fundamental mode} \]

\[ V < 2.405 \text{ or } \lambda > \lambda_c \text{ Single mode fiber} \]

\[ V > 2.405 \text{ Multimode fiber} \]

**Number of modes**

\[ M \approx \frac{V^2}{2} \]
Modes in an Optical Fiber

Normalized propagation constant

\[ b = \frac{(\beta / k)^2 - n_2^2}{n_1^2 - n_2^2} \]

\[ k = \frac{2\pi}{\lambda} \]

Normalized propagation constant \( b \) vs. \( V \)-number for a step-index fiber for various LP modes

\[ b \approx \left(1.1428 - \frac{0.996}{V}\right)^2 \quad (1.5 < V < 2.5) \]
Higher Order Modes

\[ b \approx \left(1.1428 - \frac{0.996}{V}\right)^2 \quad 1.5 < V < 2.5 \]

**Figure 2.5:** Normalized propagation constant \( b \) as a function of normalized frequency \( V \) for a few low-order fiber modes. The right scale shows the mode index \( \bar{n} \). (After Ref. [34]; ©1981 Academic Press; reprinted with permission.)
Group Velocity and Group Delay

Consider a single-mode fiber with core and cladding indices of 1.4480 and 1.4400, core radius of 3 μm, operating at 1.5 μm. Given that we can approximate the fundamental mode normalized propagation constant $b$ by Eq. (2.3.13), calculate the propagation constant $\beta$. Change the operating wavelength to $\lambda'$ by a small amount, say 0.1%, and then recalculate the new propagation constant $\beta'$. Then determine the group velocity $v_g$ of the fundamental mode at 1.5 μm, and the group delay $\tau_g$ over 1 km of fiber.

$$b \approx \left(1.1428 - \frac{0.996}{V}\right)^2 \quad 1.5 < V < 2.5$$

$$k = \frac{2\pi}{\lambda} = 4,188,790 \text{ m}^{-1} \quad \text{and} \quad \omega = \frac{2\pi c}{\lambda} = 1.255757 \times 10^{15} \text{ rad s}^{-1}$$

$$V = (2\pi a/\lambda)(n_1^2 - n_2^2)^{1/2} = 1.910088$$

$$b = 0.3860859,$$

$$b = \frac{(\beta/k) - n_2}{n_1 - n_2} \quad \Rightarrow \quad \beta = n_2 k [1 + b \Delta]$$

$$\beta = 6.044796 \times 10^6 \text{ m}^{-1}.$$
Group Velocity and Group Delay

Increase wavelength by 0.1% and recalculate. Values in the table

<table>
<thead>
<tr>
<th>Calculation →</th>
<th>$V$</th>
<th>$k$ (m$^{-1}$)</th>
<th>$\omega$ (rad s$^{-1}$)</th>
<th>$b$</th>
<th>$\beta$ (m$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda = 1.500000 \mu$m</td>
<td>1.910088</td>
<td>4188790</td>
<td>$1.255757 \times 10^{15}$</td>
<td>0.3860859</td>
<td>$6.044818 \times 10^6$</td>
</tr>
<tr>
<td>$\lambda' = 1.50150 \mu$m</td>
<td>1.908180</td>
<td>4184606</td>
<td>$1.254503 \times 10^{15}$</td>
<td>0.3854382</td>
<td>$6.038757 \times 10^6$</td>
</tr>
</tbody>
</table>

$$V_g = \frac{d\omega}{d\beta} = \frac{\omega' - \omega}{\beta' - \beta} = \frac{(1.254503 - 1.255757) \times 10^{15}}{(6.038757 - 6.044818) \times 10^6} \approx 2.07 \times 10^8 \text{ ms}^{-1}$$

The group delay $\tau_g$ over 1 km is 4.83 $\mu$s
Numerical Aperture NA

Maximum acceptance angle $\alpha_{\text{max}}$ is that which just gives total internal reflection at the core-cladding interface, i.e. when $\alpha = \alpha_{\text{max}}$ then $\theta = \theta_c$. Rays with $\alpha > \alpha_{\text{max}}$ (e.g. ray B) become refracted and penetrate the cladding and are eventually lost.

$$\sin \alpha_{\text{max}} = \frac{\left(n_1^2 - n_2^2\right)^{1/2}}{n_0} = \frac{\text{NA}}{n_o}$$

$$2\alpha_{\text{max}} = \text{total acceptance angle}$$
Example: A multimode fiber and total acceptance angle

A step index fiber has a core diameter of 100 μm and a refractive index of 1.480. The cladding has a refractive index of 1.460. Calculate the numerical aperture of the fiber, acceptance angle from air, and the number of modes sustained when the source wavelength is 850 nm.

Solution
Example: A single mode fiber

A typical single mode optical fiber has a core of diameter 8 μm and a refractive index of 1.460. The normalized index difference is 0.3%. The cladding diameter is 125 μm. Calculate the numerical aperture and the total acceptance angle of the fiber. What is the single mode cut-off frequency $\lambda_c$ of the fiber?

Solution

The numerical aperture

$$NA = (n_1^2 - n_2^2)^{1/2} = [(n_1 + n_2)(n_1 - n_2)]^{1/2}$$

Substituting $(n_1 - n_2) = n_1\Delta$ and $(n_1 + n_2) \approx 2n_1$, we get

$$NA \approx [(2n_1)(n_1\Delta)]^{1/2} = n_1(2\Delta)^{1/2} = 1.46(2 \times 0.003)^{1/2} = 0.113$$

The acceptance angle is given by

$$\sin \alpha_{max} = NA/n_o = 0.113/1 \text{ or } \alpha_{max} = 6.5^\circ, \text{ and } 2\alpha_{max} = 13^\circ$$

The condition for single mode propagation is $V \leq 2.405$ which corresponds to a minimum wavelength $\lambda_c$ is given by

$$\lambda_c = [2\pi\alpha NA]/2.405 = [(2\pi)(4 \mu m)(0.113)]/2.405 = 1.18 \mu m$$

Wavelengths shorter than 1.18 μm will result in multimode operation.
Dispersion

- Intermode (Intermodal) Dispersion: Multimode fibers only
- Materials Dispersion
  Group velocity depends on $N_g$ and hence on $\lambda$
- Waveguide Dispersion
  Group velocity depends on waveguide structure
- Chromatic Dispersion
  Material dispersion + Waveguide Dispersion
- Profile Dispersion
- Polarization Dispersion
Emitter emits a spectrum $\Delta \lambda$ of wavelengths.

Waves in the guide with different free space wavelengths travel at different group velocities due to the wavelength dependence of $n_1$. The waves arrive at the end of the fiber at different times and hence result in a broadened output pulse.

$$\frac{\Delta \tau}{L} = D_m \Delta \lambda$$

$D_m$ = Material dispersion coefficient, ps nm$^{-1}$ km$^{-1}$
Material Dispersion (SMF)

Field

Group Velocity

\[ v_g \text{(medium)} = \frac{d\omega}{dk} = \frac{c}{n - \lambda \frac{dn}{d\lambda}} \]

\[ N_g = n - \lambda \frac{dn}{d\lambda} \]
Material Dispersion (SMF)

\[ v_g = \frac{c}{N_g} \]

Group velocity Depends on the wavelength

\[ \frac{\Delta \tau}{L} = D_m \Delta \lambda \]

\[ D_m = \text{Material dispersion coefficient, ps nm}^{-1} \text{ km}^{-1} \]

\[ D_m \approx -\frac{\lambda}{c} \left( \frac{d^2 n}{d\lambda^2} \right) \]
Waveguide Dispersion (SMF)

Wavelength Dependent!!

\[ V = \frac{2\pi a}{\lambda} \left( n_1^2 - n_2^2 \right)^{1/2} \]

Normalized propagation constant

\[ b = \frac{(\beta/k)^2 - n_2^2}{n_1^2 - n_2^2} \]

\[ k = \frac{2\pi}{\lambda} \]

\[ b \approx \left( 1.1428 - \frac{0.996}{V} \right)^2 \]
Waveguide Dispersion (SMF)

\[ \frac{\Delta \tau}{L} = D_w \Delta \lambda \]

\[ D_w = \text{waveguide dispersion coefficient} \]

\[ D_w = \text{depends on the waveguide structure, ps nm}^{-1} \text{ km}^{-1} \]

\[ D_w = \frac{n_2 \Delta}{c \lambda} \left[ V \frac{d^2(bV)}{dV^2} \right] \]
Chromatic Dispersion (SMF)

Waveguide dispersion depends on the guide properties.
Chromatic Dispersion (SMF)

Material dispersion coefficient ($D_m$) for the core material (taken as SiO$_2$), waveguide dispersion coefficient ($D_w$) ($\alpha = 4.2 \ \mu m$) and the total or chromatic dispersion coefficient $D_{ch} (= D_m + D_w)$ as a function of free space wavelength, $\lambda$

Chromatic = Material + Waveguide

$$\frac{\Delta \tau}{L} = (D_m + D_w)\Delta \lambda$$