Notes on Cash Flows and Rate of Return

**Time Value of Money**

A cash flow is a series of payments or receipts spaced out in time. The key concept in analyzing cash flows is that receiving a $1 today is more desirable than receiving a $1 at some point in the future. How much more desirable $1 today is compared to $1 in the future depends on the person or firm’s point of view, of course. The person or firm, which we will simply refer to as the decision maker, quantifies what he or she thinks $1 in the future is worth by deciding what quantity of money, received today, would be equally desirable.

For example, the decision maker may believe, “I would just as soon get $0.80 today as wait a year for that dollar.” To formalize the logic, the decision maker defines a *discount factor* $\delta$ and in our example sets it equal to 0.80. It’s important to remember that this is the decision maker’s *personal* discount factor, and other people’s discount factors might be different. (For example, someone else might value $1 one year from now as much as $0.90 today and pick a discount factor of 0.90.)

If someone asked our decision maker to value $5 one year in the future, he or she would multiply $5 times the discount factor $\delta=0.80$, yielding $4. Often it is helpful to visualize the cash flow on a timeline. For example, a cash flow where $5 is received one year from now might be drawn as:

![Timeline Diagram](image.png)

The horizontal axis is in units of years. It is common convention to label the current year as “0.” Because the decision maker’s discount factor $\delta$ is 0.80, the above cash flow is exactly as desirable as the cash flow below:

![Timeline Diagram](image.png)
**Net Present Value**

The same approach can be used to compute how desirable it would be to receive $1 two years from now.

In the figure below, we multiply the $1 in year 2 by $\delta$ to determine that it would be equally desirable to receive $0.80 a year earlier, and in this case a year earlier would be year 1. Then we multiply $0.80 by the discount factor delta to determine that is equally desirable to receive $0.64, a year earlier, which would be year 0. This calculation is equivalent to taking original amount $1, and multiplying it by $\delta^2$. Thus it is equally desirable to receive $1.00 in two years as it is to receive $0.64 today. What we have computed is what is called a *Net Present Value* (*NPV*). The net present value of a cash flow is a quantity of money, which if received today, would be equally desirable as the cash flow. So the cash flow of receiving $1 in year 2, has an NPV of $\delta^2 = $0.64. Note that the answer depends on the value of our discount factor $\delta$.

We can generalize this example by computing the NPV of $1 received in year $n$. 

![Diagrams showing the computation of net present value](https://via.placeholder.com/150)
$1$ in year $n$ is as desirable as $\delta$ in year $n-1$, or as desirable as $\delta^2$ in year $n-2$, and so on. By this logic, $1$ in year $n$ is as desirable as $\delta^n$ received today. Thus, the NPV of $x_n$ received in year $n$ is $x_n \delta^n$.

A cash flow may also have payments or receipts in multiple years. Consider the following cash flow:

**Figure 1:**

To analyze the NPV of such a cash flow, one approach would be to compute the NPV of each payment or receipt, and then add them together. For example, the NPV of the $(-3)$ payment in year 0 has an NPV of – that’s right – $(-3)$: no discount factor needs to be applied because the payment happens in year 0, i.e., “the present.” The NPV of the $1$ receipt in year 1 has an NPV of $\delta$. The NPV of the $1$ receipt in year 2 has an NPV of $\delta^2$. Finally, the NPV of the $2$ receipt in year 3 has an NPV of $2 \times \delta^3$. Thus the NPV of the overall cash flow is

$$(-3) + \delta + \delta^2 + 2 \times \delta^3.$$ 

When the discount factor is set to 0.80, the above expression works out to be -$0.536. Imagine that the above cash flow were the cash flow of particular project (i.e., you plan to spend $3$ million today, get $1$ million of benefit in year 1 and 2 each, and $2$ million in year 3). The above analysis says that this project’s NPV is -$536,000 – which suggests you should not proceed with the project. However, you should remember that this is all highly dependent on the discount factor $\delta$ which you chose, and which is a reflection of how much you value money received in the future. Suppose for a moment that your discount factor was 1: how would your analysis change?
Using the same reasoning as in the above example, we can derive a more general formula. A receipt of \( x_0 \) in year 0 has an NPV of \( \delta x_0 \). A receipt of \( x_1 \) in year 1 has an NPV of \( \delta x_1 \). A receipt of \( x_2 \) in year 2 has an NPV of \( \delta^2 x_2 \). A receipt of \( x_n \) in year \( n \) has an NPV of \( \delta^n x_n \). Thus if we have a cash flow for which we receive \( x_0 \) in year 0, \( x_1 \) in year 1, \( x_2 \) in year 2, and so on, the NPV of the entire cash flow is:

\[
NPV = x_0 + \delta x_1 + \delta^2 x_2 + \delta^3 x_3 + \delta^4 x_4 + \cdots = \sum_{n=0}^{\infty} (\delta^n x_n)
\]  

(1)

**Relation of Interest Rate to Discount Factor**

Often, a decision maker’s discount factor is based on the prevailing interest rate or rate of return she could receive if she were to invest her capital in a bank or perhaps in another, alternative project\(^2\). For example, suppose the interest rate \( i \) was 0.10 (10%). Then having $1 in the bank today would be worth $1 \times (1+i) = $1.10 a year from now. Conversely, receiving $1.10 a year from now would be as desirable as having $1 today that could be put in the bank. Thus the NPV of getting $x one year from now is $x / (1+i)$. But we have also said that the NPV of getting $x one year from now is $\delta x$. Setting these two expressions to be equal to each other, we can derive the relationship between the discount factor and the interest rate

\[
\delta = \frac{1}{1+i}.
\]

(2)

If we substitute this relation between the interest rate \( i \) and the discount factor \( \delta \) into equation (1), we get an expression for NPV in terms of the interest rate \( i \):

\[
NPV = x_0 + (1+i)^{-1} x_1 + (1+i)^{-2} x_2 + (1+i)^{-3} x_3 + \cdots = \sum_{n=0}^{\infty} (1+i)^{-n} x_n
\]  

(3)

In summary, if you are given an interest rate \( i \) and asked to compute the NPV of a cash flow, you should use equation (3). If you are given a discount factor, and asked to do the same thing, you should use equation (1). If you don’t have either a discount factor or an interest rate, then you don’t have enough information to compute the NPV.

**Discount Rate**

In some problems, you may be given what is called a discount rate. The discount rate plays the same role as the interest rate in the above discussion, in that it describes by what percentage we should discount future payments. For example, if you are told that the discount rate \( i \) is 0.25 (25%), the corresponding discount factor \( \delta \) is simply

\[
\delta = \frac{1}{1+i} = \frac{1}{1+0.25} = 0.8.
\]

(2a)

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\(^1\) I’m going to drop the ‘$’ frequently from now on. It should be obvious from the context when a number is to be understood as a quantity of money, and when it’s just a number – or I’ll say so in the text.

\(^2\) The concept of a ‘bank’ here is supposed to stand for an investment of zero risk. Laugh if you wish. Risk of course will heavily influence the decision maker’s choice of discount factor, whatever the investment.
Rate of Return or Return on Investment

The return on investment (ROI), more commonly called the rate of return (ROR), is the inverse problem to computing the NPV. When one computes the ROR on an investment decision one is essentially asking, “What would the interest rate at the bank\(^3\) have to be in order for me to be neutral about investing in my project?” Thus, we are trying to find an interest rate \(i\) such that the NPV of the project is 0, as expressed by the equation:

\[
NPV = x_0 + (1 + i)^{-1}x_1 + (1 + i)^{-2}x_2 + (1 + i)^{-3}x_3 + \cdots = \sum_{n=0}^{\infty} (1 + i)^{-n}x_n = 0 \quad (3a)
\]

For example consider the cash flow

One’s intuition might immediately suggest that the ROR is 25%, but let us verify this. We want to compute an interest rate \(i\) for which we are neutral about the investment, meaning that we find some \(i\) for which the NPV is 0. We use equation (3a), to get \(-1 + (1 + i)^{-1} + 1.25 = 0\), which we can easily solve to find \(i = 0.25\). Thus the ROR is indeed 25%, which means if the interest rate at a bank were 25%, we would be neutral about proceeding with a project that required $1 investment today and would yield $1.25 in revenue tomorrow. This is because we could also invest that $1 in the bank and get exactly the same return.

Now consider the cash flow

Again we use equation (3a) to get

\(^3\)More generally, the question one asks is “What would the rate of return have to be on every other investment alternative (and one such alternative is putting the money in the bank) for me to neutral about investing in this project?”
Thus the ROR of this cash flow is also 0.25. Again suppose the cash flow represents what would happen if we invested in some project. The intuition is that if we put $1 in a bank that paid 25% interest, instead of doing the project, then in 2 years that $1 would be worth $(1.25)^2 = 1.5625$, which is the same outcome we would get if we had done the project. Thus when the interest rate is 25%, we are neutral between the alternatives of doing the project or putting the money in a bank.

Now consider the cash flow depicted below

**FIGURE 2:**

Again, suppose this cash flow is what would happen if we decided to invest in a project. We want to find the ROR, which is the interest rate for which the NPV is 0. Using equation (3a) again, we can write it as

\[-1.44 + (1 + i)^{-1}(1.00) + (1 + i)^{-2}(1.00) = 0\]
\[(1 + i)^{-2} + (1 + i)^{-1} - 1.44 = 0\]
\[(1 + i)^{-1} = \frac{-1 \pm \sqrt{1^2 + 5.76}}{2}\]
\[(1 + i)^{-1} = \frac{-1 \pm 2.6}{2}\]
\[i = 0.25 \text{ or } i = -1.5555\]

By dropping the nonsensical answer -1.5555, we find that the ROR is 0.25. Thus when the interest rate available at the bank, or in other investment opportunities, is 0.25 we are neutral about proceeding with a project that has the cash flow of Figure 2. To see further why this would be so, suppose we put $1.44 in the bank in year 0 – instead of investing in the project with the cash flow shown in Figure 2. With an interest rate of 0.25, the $1.44 would become $1.44 \times 1.25 = $1.80 in year 1. In year 1, we could withdraw $1 from the bank, and keep the remaining $0.80 in the account. In year 2, that $0.80 would grow to $0.80 \times 1.25 = $1.00, and then we
could withdraw that dollar, and by so doing, we would have duplicated the cash flow in Figure 2 – by investing in the bank instead of investing in the project.

Now suppose the interest rate at the bank were 30%. Because the ROR of the project whose cash flow is illustrated by Figure 2 is only 25%, we should prefer to invest in the bank rather than the project. This can be confirmed by an NPV analysis. If we compute the NPV using equation (3) we find that the NPV is

\[ -1.44 + (1 + 0.3)^{-1}(1.00) + (1 + 0.3)^{-2}(1.00) = -0.0791 \]

which, being negative, qualifies this project as a Bad Idea – when the interest rate is 30%.

**Infinite Series**

Sometimes a series of payments or receipts is never ending. One special case is when the same payment or receipt is made or received every year forever:

\[
\begin{array}{c}
\text{Time} \\
0 & 1 & 2 & \ldots \\
\$x & \$x & \$x & \ldots
\end{array}
\]

The NPV of this infinite series of receipts can be put into a closed form expression by the using equation (1) and making the following manipulations:

\[
x + \delta x + \delta^2 x + \delta^3 x + \cdots = NPV
\]

\[
\delta x + \delta^2 x + \delta^3 x + \cdots = \delta(NPV)
\]

Subtracting the two equations,

\[ x = (1 - \delta)NPV \]

\[
NPV = \frac{1}{1 - \delta} (x)
\]

We can even express the NPV in terms of an interest rate \( i \) (instead of the discount factor \( \delta \)) by using Equation 2:

\[
NPV = \frac{1}{1 - (1 + i)^{-1}} (x)
\]

\[
= \frac{1 + i}{i} (x) = \left(1 + \frac{1}{i}\right)(x)
\]
The above analysis assumed that the cash flow begins in year 0. If we assume instead that the cash flow begins in year 1, then we can simply subtract the NPV of the year 0 cash flow. Our cash flow in this second case is:

\[
\begin{array}{c|c|c}
0 & 1 & 2 \\
\hline
\$x & \$x & \ldots \\
\end{array}
\]

and since the year 0 cash NPV is just \( x \), we subtract that from the expression we just derived:

\[
NPV' = \left(1 + \frac{1}{i}\right) (x) - (x) = \frac{1}{i} (x)
\]