TIM 206 Lecture 4: Duality and Sensitivity

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1 Overview

1. Review Simplex Method

2. Sensitivity Analysis: How does solution change as parameters change? How much is the optimal solution affected by changing $A, b,$ or $c$? How much would we pay for more resources?


2 Simplex Method

Objective: Max $Z$ (the first line of the table)

Recall: You have a set of linear equations that can be manipulated in the usual algebraic ways.

Start with $Ax \leq b$

Convert to $(A : I)(x : x_s)^T = b$ (augmented form w/ slack variables)

Thus far we have solved problems like the following:

$$\begin{align*}
\text{Max} & \quad c^T x = Z \\
\text{s.t.} & \quad Ax \leq b \\
& \quad x \geq 0
\end{align*}$$

Goal is to increase $Z$ without violating the conditions. Note if your initial problem/constraints don’t look like this, use transformations to put it into this form. Then we need an initial basic feasible solution (corner point solution) – do this using Phase I.
Transformations

\[ Ax \geq b \iff -ax \leq -b \]
\[ Ax = b \iff ax \leq b \text{ and } ax \geq b \]
\[ x \text{ free } \iff x = x_1 - x_2 \text{ where } \]
\[ x_1 \geq 0 \text{ and } x_2 \geq 0 \]

Simplex Rules:

- update basic set of vars
- keep looking at feasible basic solutions.
- keep positive \( x_i \) values, negative values indicate non-optimality
- zeros in top line correspond to basic vars.
- slack coefficients show how you got to where you are.
- goal: top line is non-negative.

All of this assumes a well-defined initial problem. What happens when we are unsure of what the parameters should be? Might be incorrect about the cost of something and need to restart, is there an efficient way to do so?

Using the above LP, the simplex algorithm leads to:

\[
\begin{pmatrix}
1 & -c & 0 \\
0 & A & I
\end{pmatrix}
\begin{pmatrix}
z \\
x
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
b
\end{pmatrix}
\]

where \( x_s \) = slack variables.

Simplex method selects a subset to be zero, corresponding to a basis \( B \) of \( A \).

After all the simplex math, the table becomes

\[
\begin{pmatrix}
1 & c_B B^{-1} A & c_B B^{-1} \\
0 & B^{-1} A & B^{-1}
\end{pmatrix}
\begin{pmatrix}
z \\
x
\end{pmatrix}
= 
\begin{pmatrix}
c_B B^{-1} b \\
B^{-1} b
\end{pmatrix}
\]

Simplex always looks something like this where we can perform a simple substitution where \( z^* = c_B B^{-1} A, \ s^* = B^{-1}, \ y^* = C_B B^{-1} \) and \( b^* = s^* b \) resulting in
\[
\begin{pmatrix}
1 & z^* - c & y^* \\
0 & A^* & s^*
\end{pmatrix}
\begin{pmatrix}
z \\
x
x_s
\end{pmatrix} =
\begin{pmatrix}
z^* \\
b^*
\end{pmatrix}
\]

So if we know \( s^* \) and our initial parameters, we know everything about where we’ve been throughout the Simplex Method. All information about how you started is available in the tableau. \( s^* \) helps in reformulation.

An aside: always two ways of looking at the final Simplex table: slack variables versus non-slack vars, basic vs. non-basic vars. Just depends on how you decide to divide the table up.

3 Fundamental insight

Given initial table and \( y^* \) and \( s^* \) can work out rest of table.

*VIEW software example given in class, from text publisher:*

\[
\text{Max } Z = 20x_1 + 10x_2 \text{ etc...}
\]

This is helpful in sensitivity analysis. Maybe \( b \) was incorrect at the outset. Use the “new” initial table with the correct \( b \) and the already calculated \( s^* \) and \( y^* \) to see if conditions for optimality are still met. This is, of course, not limited to just the \( b \) values.

4 Sensitivity analysis

How sensitive is my solution to my assumptions?
If my assumptions were incorrect, is my solution still optimal?
Maybe increasing a parameter doesn’t even change the optimal solution. Then again, maybe it does.
Check using \( y^* \) and \( s^* \) to recalculate \( Z \). The same check for optimality still holds.

\( b \) corresponds to how much resource is available. Changing this value changes the intercept of a boundary line. If the original solution lay on that boundary, then it may have changed. Make a change in \( b \), and recalculate \( Z^* \) and \( b^* \).

\( c \) corresponds to objective coeff’s. Changes the line that we move to find the maximum. When any of these coeff’s change we may need to do some Gaussian elimination to get the table in proper form (i.e. and identity matrix somewhere). Change in this parameter never effects feasibility, though may effect optimality. Change a value in \( c \) and recalculate \( z^* - c \).

\( A \) corresponds to the cost of a resource. Changing this produces a change in the slope of one of the boundary lines, changing the feasible region. Change an \( a_{ij} \) and recalculate \( z^* - c \).
What range can you change a parameter by and keep the current optimal sol’n? Answering this type of question is called *ranging*.

### 4.1 Introducing a new variable

Same as changing the coeff’s to a non-basic variable. For example:

\[
Z = 2x_1 + 3x_2
\]

s.t. \( x_1 + x_2 \leq b_1 \)

is mathematically equivalent to the following:

\[
Z = 2x_1 + 3x_2 + 0x_3
\]

s.t. \( x_1 + x_2 + 0x_3 \leq b_1 \)

Adding the \( x_3 \) term is like having it there before, but with a coefficient of zero. Hence, we can include it and then perform the sensitivity checks as before.

### 4.2 Introducing a new constraint

The introduction of a new constraint only effects the feasible region. Furthermore, it can only make the region smaller. Therefore, if the original solution obeys the new constraint, it is still feasible. Because the goals of optimality have not changed, the solution is still optimal. If the original solution is no longer feasible, add a row to the tableau and proceed as normal.

### 4.3 Parametric LP (similar to ranging)

\[
Z = (c + \lambda d)^T x
\]

s.t. \( AX \leq b \)

\[ x \geq 0 \]

Are there values of the parameters for which the problem has a solution?

Same as solving simplex method except you may have a parameter instead of constant. Keep the parameter through the simplex method. If, at the end, the parameter appears in the top row, then this indicates constraints on that parameter as those values must be non-negative.

Max \( x_1 + 2x_2 + kx_3 \)

Why is sensitivity analysis important? May have a large program that takes several hours to solve, if you need to change an assumption, nice to check if it will have an effect on the optimal solution.
5 Duality

The Diet Problem

Min $\sum c_j x_j$ (total cost) (ie. $c^T x$)

s.t. $\sum_j a_{ij} x_j \geq b_i$ (minimal nutrients req’d) (ie. $Ax \geq b$)

$x \geq 0$

where

$c_j$ = cost to purchase food $j$

$a_{ij}$ = nutrients $i$ in food $j$

$b_i$ = req’d dose of nutrient $i$

$x_j$ = amt of food $j$ to purchase

Compare the Diet Problem with this LP:

Max $\sum b_i y_i$ (ie. $by$)

s.t. $\sum_j a_{ij} y_i \leq c_j$ (ie. $Ay \leq c$)

where

$y_i$ = price of nutrient sold

These are two perspectives on the same problem. The first can be thought of as someone buying food to meet the minimal nutrition requirements at the lowest cost and the second is the perspective of a seller trying to maximize profit while minimizing the cost to produce the goods. If the total cost for all the nutrients is $\geq c_j$ then it is more economical to buy food than the individual nutrients, i.e. $\sum_i a_{ij} \leq c_j$ for all $j$.

These two problems are called ‘primal’ and ‘dual’ respectively. They are both based on the $A, b,$ and $c$, but we rearrange them and use them differently in each LP. Note that dual(dual(primal)) = primal.

**Definition:** For the primal LP:

Max $c^T x = Z$

s.t. $AX \leq b$

$x \geq 0$

there is a dual LP:

Min $b^T y = Z^T$

s.t. $A^T y \geq c$

$y \geq 0$
Notice: Dual feasibility. If you let \( y = c^T_B B^{-1} \) where \( B \) is basis, then dual feasibility \( \iff z_j - c_j \geq 0 \), i.e. \( \implies \) primal optimality condition is satisfied.

5.1 Weak duality thm.
If \( x \) is feasible to primal and \( y \) is feasible to dual then
\[
y^T b \geq c^T x
\]
(They bound each other. The maximum primal \( \leq \) the minimum dual.)
Further, if \( y^T b = c^T x \) then \( x \) and \( y \) are optimal solutions to primal and dual, respectively.

**Proof of weak duality thm**
\[
Ax \leq b \implies \text{P-feasible}
\]
\[
y^T A \geq c^T \implies \text{D-feasible}
\]
And since \( x, y \geq 0 \)
\[
y^TAx \leq y^T b
\]
\[
y^TAx \geq c^T x
\]
Thus,
\[
c^T x \leq y^TAx \leq y^T b
\]
\[
c^T x \leq y^T b \quad (\text{Our conclusion})
\]

5.2 Strong Duality Thm.
One of the following must be true:

1. Both the primal and dual (P and D) have optimal solutions with the same objective value.
2. One problem is feasible with unbounded objective, the other is infeasible.
3. Both are infeasible.
If they are both feasible, then both have objective function that is finite.

5.3 Optimality conditions
\( x^* \) is optimal for P iff \( \exists y^* \) such that
\[
Ax \leq b_1 x \geq 0 \quad \text{P-feasible}
\]
\[
y^T A \geq c_1 y \geq 0 \quad \text{D-feasible}
\]
\[
y^T (Ax - b) = 0 \quad \text{complementary}
\]
\[
(y^T A - c^T) x = 0 \quad \text{slackness}
\]
\[
(y^T A - c^T)_i x_i = 0 \quad \text{looking for binding constraints}
\]
Note that \( y^T \) is the dual variable term, \( x \) is the primal variable term, and the terms they are being multiplied with are the primal constraints and the dual
constraints, respectively. If one constraint in $P$ is not binding, then the corresponding dual variable is zero.

Non-binding variable means that adding more of this resource doesn’t help optimization.

5.4 Dual of the dual

Definition: For the primal LP:

\[
\begin{align*}
\text{Min} & \quad b^T y \\
\text{s.t.} & \quad A^T y \geq c \\
& \quad y \geq 0
\end{align*}
\]

Translate primal LP to Standard form:

\[
\begin{align*}
\text{Max} & \quad -b^T y \\
\text{s.t.} & \quad -A^T y \leq -c \\
& \quad y \geq 0
\end{align*}
\]

Dual of the Standard form:

\[
\begin{align*}
\text{Min} & \quad -c^T z \\
\text{s.t.} & \quad (-A^T)^T z \geq (-b^T)^T \\
& \quad z \geq 0
\end{align*}
\]

Translate Dual LP to Standard form:

\[
\begin{align*}
\text{Max} & \quad c^T z \\
\text{s.t.} & \quad Az \leq b \\
& \quad z \geq 0
\end{align*}
\]

6 Dual Simplex Method

1. It is sometimes easier to solve the dual and then create a solution to the primal.

2. Can use the “Dual Simplex” method

Idea: instead of keeping $x$ feasible and checking the optimality conditions on each iteration of the simplex method, we will try to keep the optimality conditions and then check to see if $x$ is feasible. Example:

\[
\begin{array}{c|cccccc}
Z & x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
Z & 1 & 0 & 4 & 0 & 9 & 0 \\
x_3 & 0 & 0 & 3 & 1 & -1 & 0 \\
x_5 & 0 & 0 & -4 & 0 & -2 & 1 \\
x_1 & 0 & 1 & -2 & 0 & -4 & 0 \\
\end{array}
\]
Note positives on the top line, but the constraints yield negative values. This would be an optimal solution, but it is clearly infeasible. We will use the ratio test and remove the biggest violator (-12). Pivot \( x_2 \) and \( x_5 \).

\[
\begin{array}{ccccc|c}
1 & 0 & 0 & 0 & 7 & 1 & -12 \\
0 & 0 & 0 & 1 & -\frac{5}{2} & \frac{3}{2} & -15 \\
0 & 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & 3 \\
0 & 1 & 0 & 0 & -\frac{5}{3} & -\frac{1}{2} & 4
\end{array}
\]

Still infeasible, go one more step. Remove \( x_3 \), use ratio test, add \( x_4 \):

\[
\begin{array}{ccccc|c}
1 & 0 & 0 & 14 & 0 & \frac{31}{15} & -54 \\
0 & 0 & 1 & -\frac{3}{2} & 1 & -\frac{1}{2} & 6 \\
0 & 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\
0 & 0 & 1 & -\frac{1}{3} & 0 & -\frac{1}{2} & 22
\end{array}
\]

Now we see all positive coeffs on the top row, thus optimal. Plus we have all positives in the constraint column, thus feasible. So we are done. This is analogous to simplex method but in a dual spaces.