Linear Programming
Simplex in Matrix Form
and the Fundamental insight

James G. Shanahan
Independent Consultant
and UC Santa Cruz
EMAIL: James_DOT_Shanahan_AT_gmail_DOT_com

WIFI: SSID Student
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TIM 206 (30155)  Introduction to Optimization Theory and Applications
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Lecture 03

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Course Info: Solutions

- [http://courses.soe.ucsc.edu/courses/tim206](http://courses.soe.ucsc.edu/courses/tim206)
- [https://courses.soe.ucsc.edu/courses/tim206/Winter13/01](https://courses.soe.ucsc.edu/courses/tim206/Winter13/01)
  - Schedule
  - Exam during
Performance Evaluation

Final Exam (closed book):
   Week 11 of the Quarter

Performance Evaluation:
Homework     30%
Midterm      20% (Week 6 of the Quarter)
Class participation 20%
Final Exam    30%   (Week 11 of the Quarter)
Audience Participation
Reading Material

• Chapter 4 and 5 from H&L Book
LP Lecture 3 Schedule

• **Last lecture**
  – Introduction and background material
  – Properties of LPs
  – Simplex method via geometry and algebraically, via tableaus

• **This lecture**
  – Adapting simplex to other forms (\(=, \geq\), negative \(b\))
  – Two Phase, Big-M and Artificial Variable technique.
  – Sensitivity Analysis
  – Shadow Prices
  – Simplex via matrices
  – Fundamental insight

• **Next Lecture**
  – Sensitivity and Duality
  – Alternative Methods for solving
Simplex Tableau Method: Init

- Introduce slack variables. Select the decision variables to be the initial nonbasic variables (set equal to zero) and the slack variables to be the initial basic variables.
  - (See Sec. 4.6 for the necessary adjustments if the model is not in our standard form—maximization, only $\leq$ functional constraints, and all nonnegativity constraints—or if any bi values are negative.)

<table>
<thead>
<tr>
<th>TABLE 4.3 Initial system of equations for the Wyndor Glass Co. problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Algebraic Form</td>
</tr>
<tr>
<td>Basic Variable</td>
</tr>
<tr>
<td>----------------</td>
</tr>
<tr>
<td>$Z - 3x_1 - 5x_2$</td>
</tr>
<tr>
<td>$x_1 + x_3$</td>
</tr>
<tr>
<td>$2x_2 + x_4$</td>
</tr>
<tr>
<td>$3x_1 + 2x_2 + x_5$</td>
</tr>
</tbody>
</table>
Optimality Tests

- The current BF solution is optimal if and only if every coefficient in row 0 is nonnegative ( >= 0).
- If it is,
  - stop;
  - otherwise, go to an iteration to obtain the next BF solution, which involves changing one nonbasic variable to a basic variable (step 1) and vice versa (step 2) and then solving for the new solution (step 3).

### TABLE 4.3 Initial system of equations for the Wyndor Glass Co. problem

<table>
<thead>
<tr>
<th>(a) Algebraic Form</th>
<th>(b) Tabular Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0) <strong>Z</strong> - 3x₁ - 5x₂ = 0</td>
<td><strong>Z</strong></td>
</tr>
<tr>
<td>(1) x₁ + x₃ = 4</td>
<td>x₃</td>
</tr>
<tr>
<td>(2) 2x₂ + x₄ = 12</td>
<td>x₄</td>
</tr>
<tr>
<td>(3) 3x₁ + 2x₂ + x₅ = 18</td>
<td>x₅</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Coefficient of:</th>
<th>Z</th>
<th>x₁</th>
<th>x₂</th>
<th>x₃</th>
<th>x₄</th>
<th>x₅</th>
<th>Right Side</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0)</td>
<td>1</td>
<td>-3</td>
<td>-5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(1)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>(2)</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>(3)</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>18</td>
</tr>
</tbody>
</table>
TIE BREAKING IN THE SIMPLEX METHOD

- **Tie for the Entering Basic Variable**
  - The answer is that the selection between these contenders may be made arbitrarily. The optimal solution will be reached eventually, regardless of the tied variable chosen.

- **Tie for the Leaving Basic Variable—Degeneracy**
  - Rarely occurs in practice.
  - First, all the tied basic variables reach zero simultaneously as the entering basic variable is increased. Therefore, the one or ones not chosen to be the leaving basic variable also will have a value of zero in the new BF solution. (Note that basic variables with a value of zero are called degenerate, and the same term is applied to the corresponding BF solution.)
  - \( Z \) may remain the same rather than increase at each iteration, the simplex method may then go around in a loop.
  - If a loop were to occur, one could always get out of it by changing the choice of the leaving basic variable.
No Leaving Variable (Unbounded Z)

• In step 2 of a simplex iteration, there is one other possible outcome that we have not yet discussed, namely, that no variable qualifies to be the leaving basic variable.

• This outcome would occur if the entering basic variable could be increased indefinitely without giving negative values to any of the current basic variables.

• In tabular form, this means that every coefficient in the pivot column (excluding row 0) is either negative or zero.
No Optimal Solutions
Unbounded Objective (unbounded $Z$)

This occurs only if:

- (1) it has no feasible solutions or
- (2) the constraints do not prevent improving the value of the objective function ($Z$) indefinitely in the favorable direction (positive or negative). The latter case is referred to as having an unbounded $Z$.

E.g., Drop last two functional constraints in WynDor Problem
Bug in model?

• The interpretation of a tableau like the one shown in Table 4.9 is that the constraints do not prevent the value of the objective function $Z$ increasing indefinitely, so the simplex method would stop with the message that $Z$ is unbounded.

• Alternatively, a computational mistake may have occurred.

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Eq.</th>
<th>$Z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>Right Side</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>(0)</td>
<td>1</td>
<td>-3</td>
<td>-5</td>
<td>0</td>
<td>0</td>
<td>None</td>
</tr>
<tr>
<td>$x_3$</td>
<td>(1)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>None</td>
</tr>
</tbody>
</table>

With $x_1 = 0$ and $x_2$ increasing, $x_3 = 4 - 1x_1 - 0x_2 = 4 > 0$. 

Multiple optimal solutions

### Table 4.10: Complete set of simplex tableaux to obtain all optimal BF solutions for the Wyndor Glass Co. problem with \(c_2 = 2\)

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basic Variable</th>
<th>Eq.</th>
<th>Coefficient of:</th>
<th>Right Side</th>
<th>Solution Optimal?</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(Z)</td>
<td>(0)</td>
<td>1 -3 -2 0 0 0 0</td>
<td>0</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>(x_3)</td>
<td>(1)</td>
<td>0 1 0 1 0 0</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(x_4)</td>
<td>(2)</td>
<td>0 0 2 0 1 0</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(x_5)</td>
<td>(3)</td>
<td>0 3 2 0 0 1</td>
<td>18</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>(Z)</td>
<td>(0)</td>
<td>1 0 -2 3 0 0</td>
<td>12</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>(x_1)</td>
<td>(1)</td>
<td>0 1 0 1 0 0</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(x_4)</td>
<td>(2)</td>
<td>0 2 0 1 0 0</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(x_5)</td>
<td>(3)</td>
<td>0 0 2 -3 0 1</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(Z)</td>
<td>(0)</td>
<td>1 0 0 0 0 0 1</td>
<td>18</td>
<td>Yes</td>
</tr>
<tr>
<td></td>
<td>(x_1)</td>
<td>(1)</td>
<td>0 1 0 1 0 0</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(x_4)</td>
<td>(2)</td>
<td>0 0 3 1 -1</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(x_2)</td>
<td>(3)</td>
<td>0 0 1 -3 0 1/2</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Extra</td>
<td>(Z)</td>
<td>(0)</td>
<td>1 0 0 0 0 0 1</td>
<td>18</td>
<td>Yes</td>
</tr>
<tr>
<td></td>
<td>(x_1)</td>
<td>(1)</td>
<td>0 1 0 0 (1/3)</td>
<td>(1/3)</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>(x_3)</td>
<td>(2)</td>
<td>0 0 0 1 (1/3)</td>
<td>(1/3)</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>(x_2)</td>
<td>(3)</td>
<td>0 0 1 0 (1/2)</td>
<td>0</td>
<td>6</td>
</tr>
</tbody>
</table>

Final Iteration
Multiple Optimal Values

Maximize \( Z = 3x_1 + 2x_2 \),
subject to \( x_1 \leq 4 \)
\( 2x_2 \leq 12 \)
\( 3x_1 + 2x_2 \leq 18 \)
and \( x_1 \geq 0, \ x_2 \geq 0 \)

Every point on this darker line segment is optimal, each with \( Z = 18 \).
Beyond Standard Form Models

• So far, focused on Maximizing $Z$ subject to functional constraints in $\leq$ form and nonnegativity constraints on all variables) and that $b_i \geq 0$ for all $i \ 1, 2, \ldots, m$

• In this section we point out how to make the adjustments required for other legitimate forms of the linear programming model.

• You will see that all these adjustments can be made during the initialization, so the rest of the simplex method can then be applied just as you have learned it already.
Identifying an initial BF Solution

- The only serious problem introduced by the other forms for functional constraints (the \( = \) or \( \geq \) forms, or having a negative right-hand side) lies in identifying an initial BF solution.
- Before, this initial solution was found very conveniently by letting the slack variables be the initial basic variables, so that each one just equals the nonnegative right-hand side of its equation.
- The standard approach that is used for all these cases is the artificial-variable technique. This technique constructs a more convenient artificial problem by introducing a dummy variable (called an artificial variable) into each constraint that needs one.
Modify Objective Function

- The objective function also is modified to impose an exorbitant penalty on their having values larger than zero.
- The iterations of the simplex method then automatically force the artificial variables to disappear (become zero), one at a time, until they are all gone, after which the real problem is solved.
Example with equality constraint

Maximize \[ Z = 3x_1 + 5x_2, \]
subject to \[ x_1 \leq 4, \]
\[ 2x_2 \leq 12, \]
\[ 3x_1 + 2x_2 = 18, \]
and \[ x_1 \geq 0, \quad x_2 \geq 0. \]

(0) \[ Z - 3x_1 - 5x_2 = 0, \]
(1) \[ x_1 + x_3 = 4, \]
(2) \[ 2x_2 + x_4 = 12, \]
(3) \[ 3x_1 + 2x_2 = 18. \]
From $=$ to $\leq$ and $-M$ in the objective

Artificial-variable Technique

Obtaining an Initial BF Solution. The procedure is to construct an artificial problem that has the same optimal solution as the real problem by making two modifications of the real problem.

1. Apply the artificial-variable technique by introducing a nonnegative artificial variable (call it $\bar{x}_5$) into Eq. (3), just as if it were a slack variable

   $$(3) \quad 3x_1 + 2x_2 + \bar{x}_5 = 18.$$  

2. Assign an overwhelming penalty to having $\bar{x}_5 > 0$ by changing the objective function $Z = 3x_1 + 5x_2$ to $Z = 3x_1 + 5x_2 - M\bar{x}_5,$

where $M$ symbolically represents a huge positive number. (This method of forcing $\bar{x}_5$ to be $\bar{x}_5 = 0$ in the optimal solution is called the Big $M$ method.)

<table>
<thead>
<tr>
<th>The Artificial Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Define $\bar{x}_5 = 18 - 3x_1 - 2x_2.$</td>
</tr>
<tr>
<td>Maximize $Z = 3x_1 + 5x_2,$</td>
</tr>
<tr>
<td>subject to $x_1 \leq 4$</td>
</tr>
<tr>
<td>$2x_2 \leq 12$</td>
</tr>
<tr>
<td>$3x_1 + 2x_2 = 18$</td>
</tr>
<tr>
<td>$x_1 \geq 0, \quad x_2 \geq 0.$</td>
</tr>
<tr>
<td>(so $3x_1 + 2x_2 + \bar{x}_5 = 18$)</td>
</tr>
<tr>
<td>and $x_1 \geq 0, \quad x_2 \geq 0, \quad \bar{x}_5 \geq 0.$</td>
</tr>
</tbody>
</table>
Phase 1: Apply Simplex to Artificial Problem

• Apply the simplex method to the artificial problem, starting with the following initial BF solution

\[\text{Initial BF Solution}\]
\[
\begin{align*}
\text{Nonbasic variables: } & \quad x_1 = 0, \quad x_2 = 0 \\
\text{Basic variables: } & \quad x_3 = 4, \quad x_4 = 12, \quad \bar{x}_5 = 18.
\end{align*}
\]
Phase 1 Solution Path

Define $\bar{x}_5 = 18 - 3x_1 - 2x_2$.
Maximize $Z = 3x_1 + 5x_2 - M\bar{x}_5$, subject to

- $x_1 \leq 4$
- $2x_2 \leq 12$
- $3x_1 + 2x_2 \leq 18$

and $x_1 \geq 0, \ x_2 \geq 0, \ \bar{x}_5 \geq 0$
Converting Equation (0) to Proper Form

- system of equations after the artificial problem is augmented

\[
\begin{align*}
(0) & \quad Z - 3x_1 - 5x_2 + Mx_5 = 0 \\
(1) & \quad x_1 + x_3 = 4 \\
(2) & \quad 2x_2 + x_4 = 12 \\
(3) & \quad 3x_1 + 2x_2 + x_5 = 18
\end{align*}
\]
Converting Equation (0) to Proper Form

- system of equations after the artificial problem is augmented

\[
\begin{align*}
(0) \quad Z - 3x_1 - 5x_2 + M\bar{x}_5 &= 0 \\
(1) \quad x_1 + x_3 &= 4 \\
(2) \quad 2x_2 + x_4 &= 12 \\
(3) \quad 3x_1 + 2x_2 + \bar{x}_5 &= 18
\end{align*}
\]

\[
\begin{align*}
Z - 3x_1 - 5x_2 + M\bar{x}_5 &= 0 \\
- M(3x_1 + 2x_2 + \bar{x}_5 = 18) \\
\text{New (0)} \quad Z - (3M + 3)x_1 -(2M + 5)x_2 &= -18M.
\end{align*}
\]

\[
Z = -18M + (3M + 3)x_1 + (2M + 5)x_2.
\]
• Since $3M + 3 \geq 2M + 5$ (remember that $M$ represents a huge number), increasing $x_1$ increases $Z$ at a faster rate than increasing $x_2$ does, so $x_1$ is chosen as the entering basic variable.

• This leads to the move from $(0, 0)$ to $(4, 0)$ at iteration 1, thereby increasing $Z$ by $4(3M + 3)$.

• The quantities involving $M$ never appear in the system of equations except for Eq. (0), so they need to be taken into account only in the optimality test and when an entering basic variable is determined.
### Phase 1 of the two-phase method

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basic Variable</th>
<th>Eq.</th>
<th>Coefficient of:</th>
<th>Right Side</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>(0)</td>
<td>$Z$</td>
<td>$1$</td>
</tr>
<tr>
<td></td>
<td>$Z$</td>
<td>(1)</td>
<td>$-3M - 3$</td>
<td>$1$</td>
</tr>
<tr>
<td></td>
<td>$x_3$</td>
<td>(2)</td>
<td>$-2M - 5$</td>
<td>$0$</td>
</tr>
<tr>
<td></td>
<td>$x_4$</td>
<td>(3)</td>
<td>$0$</td>
<td>$4$</td>
</tr>
<tr>
<td></td>
<td>$x_5$</td>
<td></td>
<td>$0$</td>
<td>$12$</td>
</tr>
</tbody>
</table>

$x_5$ is a basic variable ($x_5 > 0$) in the first two tableaux.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basic Variable</th>
<th>Eq.</th>
<th>Coefficient of:</th>
<th>Right Side</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>(0)</td>
<td>$Z$</td>
<td>$1$</td>
</tr>
<tr>
<td></td>
<td>$Z$</td>
<td>(1)</td>
<td>$-2M - 5$</td>
<td>$1$</td>
</tr>
<tr>
<td></td>
<td>$x_1$</td>
<td>(2)</td>
<td>$3M + 3$</td>
<td>$0$</td>
</tr>
<tr>
<td></td>
<td>$x_4$</td>
<td>(3)</td>
<td>$0$</td>
<td>$4$</td>
</tr>
<tr>
<td></td>
<td>$x_5$</td>
<td></td>
<td>$0$</td>
<td>$12$</td>
</tr>
</tbody>
</table>

$x_5$ is a nonbasic variable ($x_5 = 0$) in the last two (so eq constraint is satisfied).
Phase 2: of the two-phase method

• The objective for phase 2 is to find an optimal solution for the real problem. Since the artificial variables are not part of the real problem, these variables can now be dropped (they are all zero now anyway).

• Starting from the BF solution obtained at the end of phase 1, use the simplex method to solve the real problem.
  – Drop artificial variables; substitute phase 2 objective with
  – Restore proper form from Gaussian elimination so we can read off the initial BF solution
Homework Practice

• Complete solution to phase 2 for this problem
Summary of the Two-Phase Method

Summary of the Two-Phase Method.  *Initialization:* Revise the constraints of the original problem by introducing artificial variables as needed to obtain an obvious initial BF solution for the artificial problem.

*Phase 1:* The objective for this phase is to find a BF solution for the real problem. To do this,

Minimize $Z = \Sigma$ artificial variables, subject to revised constraints.

The optimal solution obtained for this problem (with $Z = 0$) will be a BF solution for the real problem.

*Phase 2:* The objective for this phase is to find an optimal solution for the real problem. Since the artificial variables are not part of the real problem, these variables can now be dropped (they are all zero now anyway). Starting from the BF solution obtained at the end of phase 1, use the simplex method to solve the real problem.
LP Healthcare Example:
Design of External Beam Radiation Therapy

- MARY has just been diagnosed as having a cancer at a fairly advanced stage. Specifically, she has a large malignant tumor in the bladder area (a “whole bladder lesion”).
- The goal of the design is to select the combination of beams to be used, and the intensity of each one, to generate the best possible dose distribution.
- Cross section of Mary’s tumor (viewed from above)
  - as well as nearby critical tissues to avoid and
  - the radiation beams being used.
  - normally dozens of possible beams must be considered
- Location of her tumor is in a tricky spot.
Design of Radiation Therapy is Key

• Because of the need to carefully balance all these factors, the design of radiation therapy is a very delicate process.

• The goal of the design
  – is to select the combination of beams to be used, and
  – the intensity of each one, to generate the best possible dose distribution.
  – (The dose strength at any point in the body is measured in units called kilorads.)

• Once the treatment design has been developed, it is administered in many installments, spread over several weeks.
For any proposed beam of given intensity, the analysis of what the resulting radiation absorption by various parts of the body would be requires a complicated process.

In brief, based on careful anatomical analysis, the energy distribution within the twodimensional cross section of the tissue can be plotted on an isodose map, where the contour lines represent the dose strength as a percentage of the dose strength at the entry point.

A fine grid then is placed over the isodose map. By summing the radiation absorbed in the squares containing each type of tissue, the average dose that is absorbed by the tumor, healthy anatomy, and critical tissues can be calculated.

With more than one beam (administered sequentially), the radiation absorption is additive (i.e., no cross product terms).
F(Beam intensity) = Absorption

* IsoDose Map

- The contour lines represent the dose strength as a percentage of the dose strength at the entry point. A fine grid then is placed over the isodose map. By summing the radiation absorbed in the squares containing each type of tissue, the average dose that is absorbed by the tumor, healthy anatomy, and critical tissues can be calculated. (measured in Kilorads)
Decide the dosage levels for each beam

- Assume two beams here (usually many more). The two decision variables $x_1$ and $x_2$ represent the dose (in kilorads) at the entry point for beam 1 and beam 2, respectively.
- Because the total dosage reaching the healthy anatomy is to be minimized, let $Z$ denote this quantity.

<table>
<thead>
<tr>
<th>Area</th>
<th>Fraction of Entry Dose Absorbed by Area (Average)</th>
<th>Restriction on Total Average Dosage, Kilorads</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Beam 1</td>
<td>Beam 2</td>
</tr>
<tr>
<td>Healthy anatomy</td>
<td>0.4</td>
<td>0.5</td>
</tr>
<tr>
<td>Critical tissues</td>
<td>0.3</td>
<td>0.1</td>
</tr>
<tr>
<td>Tumor region</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>Center of tumor</td>
<td>0.6</td>
<td>0.4</td>
</tr>
</tbody>
</table>
Formulate Beam Dosage as an LP

• Normally have 1000s of beams or more

\[
\begin{align*}
\text{Minimize} \quad & Z = 0.4x_1 + 0.5x_2, \\
\text{subject to} \quad & 0.3x_1 + 0.1x_2 \leq 2.7, \\
& 0.5x_1 + 0.5x_2 = 6, \\
& 0.6x_1 + 0.4x_2 \geq 6, \\
\text{and} \quad & x_1 \geq 0, \quad x_2 \geq 0.
\end{align*}
\]

See here for more details

The optimal design is to use a total dose at the entry point of 7.5 kilorads for beam 1 and 4.5 kilorads for beam 2.
Phase 1 Problem (Radiation Therapy Example):

Minimize \( Z = \bar{x}_4 + \bar{x}_6 \),

subject to

\[
\begin{align*}
0.3x_1 + 0.1x_2 + x_3 &= 2.7 \\
0.5x_1 + 0.5x_2 + \bar{x}_4 &= 6 \\
0.6x_1 + 0.4x_2 - x_5 + \bar{x}_6 &= 6
\end{align*}
\]

and

\[
\begin{align*}
x_1 &\geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad \bar{x}_4 \geq 0, \quad x_5 \geq 0, \quad \bar{x}_6 \geq 0.
\end{align*}
\]

Phase 2 Problem (Radiation Therapy Example):

Minimize \( Z = 0.4x_1 + 0.5x_2 \),

subject to

\[
\begin{align*}
0.3x_1 + 0.1x_2 + x_3 &= 2.7 \\
0.5x_1 + 0.5x_2 &= 6 \\
0.6x_1 + 0.4x_2 - x_5 &= 6
\end{align*}
\]

and

\[
\begin{align*}
x_1 &\geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_5 \geq 0.
\end{align*}
\]
Functional Constraints in $\geq$ Form

Radiation Therapy Example

Minimize $Z = 0.4x_1 + 0.5x_2,$
subject to
$0.3x_1 + 0.1x_2 \leq 2.7$
$0.5x_1 + 0.5x_2 = 6$
$0.6x_1 + 0.4x_2 \geq 6$
and
$x_1 \geq 0, \quad x_2 \geq 0.$
Surplus variable and artificial variable

Radiation Therapy Example

Minimize \[ Z = 0.4x_1 + 0.5x_2, \]
subject to
\[ 0.3x_1 + 0.1x_2 \leq 2.7 \]
\[ 0.5x_1 + 0.5x_2 = 6 \]
\[ 0.6x_1 + 0.4x_2 \geq 6 \]
and
\[ x_1 \geq 0, \quad x_2 \geq 0. \]

\[ \rightarrow \quad 0.6x_1 + 0.4x_2 \geq 6 \]
\[ \rightarrow \quad 0.6x_1 + 0.4x_2 - x_5 + x_6 = 6 \quad (x_5 \geq 0, \quad x_6 \geq 0). \]

Minimize \[ Z = 0.4x_1 + 0.5x_2 + M\bar{x}_4 + M\bar{x}_6, \]
subject to
\[ 0.3x_1 + 0.1x_2 + x_3 = 2.7 \]
\[ 0.5x_1 + 0.5x_2 + \bar{x}_4 = 6 \]
\[ 0.6x_1 + 0.4x_2 - x_5 + \bar{x}_6 = 6 \]
and
\[ x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad \bar{x}_4 \geq 0, \quad x_5 \geq 0, \quad \bar{x}_6 \geq 0. \]
Constraints for the artificial problem:

\[ 0.3x_1 + 0.1x_2 \leq 2.7 \]
\[ 0.5x_1 + 0.5x_2 \leq 6 \quad (= \text{holds when } x_4 = 0) \]
\[ (0.6x_1 + 0.4x_2 \geq 6 \quad \text{when } x_6 = 0) \]
\[ x_1 \geq 0, \ x_2 \geq 0 \quad (\bar{x}_4 \geq 0, \ \bar{x}_6 \geq 0) \]
Big $M$ Method:

Minimize  \[ Z = 0.4x_1 + 0.5x_2 + M\bar{x}_4 + M\bar{x}_6. \]

Two-Phase Method:

Phase 1:  Minimize  \[ Z = \bar{x}_4 + \bar{x}_6. \]

Phase 2:  Minimize  \[ Z = 0.4x_1 + 0.5x_2. \]
<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basic Variable</th>
<th>Eq.</th>
<th>Coefficient of:</th>
<th>Right Side</th>
</tr>
</thead>
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<tr>
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<td>(0)</td>
<td>-1</td>
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<tr>
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### TABLE 4.14 Preparing to begin phase 2 for the radiation therapy example

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<th>Coefficient of:</th>
<th>Right Side</th>
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<td>Final Phase 1</td>
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</tr>
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<td>Substitute phase 2</td>
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<tr>
<td>$x_1$</td>
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<td>0</td>
<td>1</td>
</tr>
<tr>
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<td>0</td>
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</tbody>
</table>
### TABLE 4.15 Phase 2 of the two-phase method for the radiation therapy example

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basic Variable</th>
<th>Eq.</th>
<th>Coefficient of:</th>
<th>Right Side</th>
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</thead>
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<td>( x_1 )</td>
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<td>0</td>
</tr>
<tr>
<td></td>
<td>( x_1 )</td>
<td>(1)</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>( x_3 )</td>
<td>(2)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>( x_2 )</td>
<td>(3)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>( Z )</td>
<td>(0)</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>( x_1 )</td>
<td>(1)</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>( x_5 )</td>
<td>(2)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>( x_2 )</td>
<td>(3)</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
FIGURE 4.7
This graph shows the sequence of CPF solutions for phase 1 (0, 1, 2, 3) and then for phase 2 (0, 1) when the two-phase method is applied to the radiation therapy example.
Postoptimality Analysis

- Postoptimality analysis—the analysis done after an optimal solution is obtained for the initial version of the model

<table>
<thead>
<tr>
<th>Task</th>
<th>Purpose</th>
<th>Technique</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model debugging</td>
<td>Find errors and weaknesses in model</td>
<td>Reoptimization</td>
</tr>
<tr>
<td>Model validation</td>
<td>Demonstrate validity of final model</td>
<td>See Sec. 2.4</td>
</tr>
<tr>
<td>Final managerial decisions on resource</td>
<td>Make appropriate division of organizational resources between activities</td>
<td>Shadow prices</td>
</tr>
<tr>
<td>allocations (the ( b_i ) values)</td>
<td>and other important activities</td>
<td></td>
</tr>
<tr>
<td>Evaluate estimates of model parameters</td>
<td>Determine crucial estimates that may affect optimal solution for</td>
<td>Sensitivity analysis</td>
</tr>
<tr>
<td>Evaluate trade-offs between model parameters</td>
<td>further study</td>
<td>Parametric linear</td>
</tr>
<tr>
<td></td>
<td>Determine best trade-off</td>
<td>programming</td>
</tr>
</tbody>
</table>
Decision Variables vs. Parameters

- Use Linear Programming as an example
  - Define problem
  - Gather data
  - Formulate model
  - Solve

Maximize $Z = 3X_1 + 5X_2$
Subject to:

1. $X_1 \leq 4$
2. $2X_2 \leq 12$
3. $3X_1 + 2X_2 \leq 18$
4. $X_1 = 2, X_2 = 6$
Reoptimization

• **Reapply the simplex method from scratch is expensive**
  – for each new version of the model, even though each run may require hundreds or even thousands of iterations for large problems.

• **A much more efficient approach is to reoptimize**
Reoptimization

- Reoptimization involves deducing how changes in the model get carried along to the final simplex tableau (as described in Secs. 5.3 and 6.6).
- This revised tableau and the optimal solution for the prior model are then used as the initial tableau and the initial basic solution for solving the new model.
- **CASE 1:**
  - If this solution is feasible for the new model, then the simplex method is applied in the usual way, starting from this initial BF solution.
- **CASE 2**
  - If the solution is not feasible, a related algorithm called the dual simplex method (described in Sec. 7.1) probably can be applied to find the new optimal solution, starting from this initial basic solution.
- **Case 3**
  - The one requirement for using the dual simplex method here is that the optimality test is still passed when applied to row 0 of the revised final tableau. If not, then still another algorithm called the primal-dual method can be used instead.
Reoptimization

• The big advantage of this reoptimization technique over re-solving from scratch is that an optimal solution for the revised model probably is going to be much closer to the prior optimal solution than to an initial BF solution constructed in the usual way for the simplex method.

• Requires only zero or a very small number of iterations
Shadow Prices

• The shadow price for resource i (denoted by $y^*_i$) measures the marginal value of this resource, i.e., the rate at which $Z$ could be increased by (slightly) increasing the amount of this resource ($b_i$) being made available.

• The simplex method identifies this shadow price by $y^*_i$ coefficient of the $i$th slack variable in row 0 of the final simplex tableau.
Example

- Information on the economic contribution of the resources to the measure of performance \( Z \) for the current study often would be extremely useful.
- The simplex method provides this information in the form of shadow prices for the respective resources.

Resource \( i \) = production capacity of Plant \( i \) \((i = 1, 2, 3) \) being made available to the two new products under consideration,

\[ b_i = \text{hours of production time per week being made available in Plant } i \text{ for these new products.} \]
Manager explores different level of resources

• **The tentative initial decision has been**
  – \( b_1 = 4, b_2 = 12, b_3 = 18, \)

• **Shadow prices**
  – \( y_1^* = 0 \) shadow price for resource 1,
  – \( y_2^* = 1.5 \) shadow price for resource 2,
  – \( Y_3 = 1 \) shadow price for resource 3.
### Shadow Prices

**TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem**

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basic Variable</th>
<th>Eq.</th>
<th>Coefficient of:</th>
<th>Right Side</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Z</td>
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<td>$x_2$</td>
</tr>
<tr>
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<td>$Z$</td>
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<td>-3</td>
</tr>
<tr>
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<td>1</td>
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<tr>
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<tr>
<td>3</td>
<td>$x_5$</td>
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<tr>
<td></td>
<td>$x_1$</td>
<td>(3)</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Define $\bar{x}_5 = 18 - 3x_1 - 2x_2$, Maximize $Z = 3x_1 + 5x_2 - M\bar{x}_5$, subject to $x_1 \leq 4$, $x_2 \leq 12$, $3x_1 + 2x_2 \leq 18$, and $x_1 \geq 0$, $x_2 \geq 0$, $\bar{x}_5 \geq 0$. 

Feasible region: (0, 6), (2, 6), (4, 3), (4, 0), (0, 0).
Shadow Prices Graphically

The optimal solution, (2, 6) with \( Z = 36 \), changes to \( (5/3, 13/2) \) with \( Z = 37.5 \) when \( b_2 \) is increased by 1 (from 12 to 13) so that:

\[
y_2^* = \Delta Z = 37.5 - 36 = 3/2
\]
Shadow Prices: Focus on $y_1^*$

**TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem**

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basic Variable</th>
<th>Eq.</th>
<th>Coefficient of:</th>
<th>Right Side</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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</table>

Define $\bar{x}_5 = 18 - 3x_1 - 2x_2$.
Maximize $Z = 3x_1 + 5x_2 - M\bar{x}_5$ subject to

$x_1 \leq 4$

$2x_2 \leq 12$

$3x_1 + 2x_2 \leq 18$

and $x_1 \geq 0$, $x_2 \geq 0$, $\bar{x}_5 \geq 0$
Shadow Prices Graphically: $y_1^*$

Should this actually be done?

Why is

$$y_0^* = \Delta Z = 36 - 36 = 3/2$$
Shadow Prices Graphically:

$y_1^* = 0$ means what?
What about constraint 3? Binding?

Because the limited supply of these resources \((b_2 = 12, b_3 = 18)\) binds \(Z\) from being increased further, they have positive shadow prices. Economists refer to such resources as scarce goods whereas resources available in surplus (such as resource 1) are free goods (resources with a zero shadow price). Discuss shadow prices more later.
Sensitivity Analysis: RHS $b_i$

- LP($a_{ij}$, $b_i$, $c_j$)
- Sensitive parameters (i.e., those that cannot be changed without changing the optimal solution)
- Case $b_i$, if $y_i^* > 0$ (non-basic) implies the optimal solution changes if $b_i$ changes ($b_i$ is a sensitive parameter; feasible region expands/contracts)
- Case $b_i$, if $y_i^* = 0$ (basic): not sensitive (to at least small changes in $b_i$)
- Pay attention to resources with large shadow prices
Sensitivity Analysis: Cost efficient \( c_j \)

- \( \text{LP}(a_{ij}, b_i, c_j) \)

With \( c_2 = 5 \), the allowable range for \( c_1 \) is \( 0 \leq c_1 \leq 7.5 \).

With \( c_1 = 3 \), the allowable range for \( c_2 \) is \( c_2 \geq 2 \).
Sensitivity Analysis: efficient \( a_{ij} \)

- The easiest way to analyze the sensitivity of each of the \( a_{ij} \) parameters graphically is to check whether the corresponding constraint is binding at the optimal solution. Because \( x_1 \leq 4 \) is not a binding constraint, any sufficiently small change in its coefficients (\( a_{11} = 1, a_{12} = 0 \)) is not going to change the optimal solution, so these are not sensitive parameters. On the other hand, both \( 2x_2 \leq 12 \) and \( 3x_1 + 2x_2 \leq 18 \) are binding constraints, so changing any one of their coefficients (\( a_{21} = 0, a_{22} = 2, a_{31} = 3, a_{32} = 2 \)) is going to change the optimal solution, and therefore these are sensitive parameters.

- Change feasible region \( \Rightarrow \) change optimal solution and \( Z \).
Allowable Range $c_j$

- For any $c_j$, its allowable range to stay optimal is the range of values for this coefficient over which the current optimal solution remains optimal, assuming no change in the other coefficients.

- When the upper table in the sensitivity report generated by the Excel Solver indicates that both the allowable increase and the allowable decrease are greater than zero for every objective coefficient, this is a signpost that the optimal solution in the “Final Value” column is the only optimal solution.

- Conversely, having any allowable increase or allowable decrease equal to zero is a signpost that there are multiple optimal solutions. Changing the corresponding coefficient a tiny amount beyond the zero allowed and re-solving provides another optimal CPF solution for the original model.
Allowable Range $b_i$

- For any $b_i$, its allowable range to stay feasible is the range of values for this right-hand side over which the current optimal BF solution (with adjusted values for the basic variables) remains feasible, assuming no change in the other right-hand sides.
More complex sensitivity analysis

• Requires using the fundamental insight described (see shortly) to deduce the changes that get carried along to the final simplex tableau as a result of changing the value of a parameter in the original model.

• (The rest of the procedure is described and illustrated in Secs. 6.6 and 6.7 of H&L)
Sensitivity analysis in Excel
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<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
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<td>6</td>
<td><strong>Hours Used Per Batch Produced</strong></td>
<td></td>
<td></td>
<td>Hours</td>
<td></td>
<td></td>
<td>Hours</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>Plant 1</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>&lt;=</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>Plant 2</td>
<td>0</td>
<td>2</td>
<td>12</td>
<td>&lt;=</td>
<td>12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>Plant 3</td>
<td>3</td>
<td>2</td>
<td>18</td>
<td>&lt;=</td>
<td>18</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>Doors</td>
<td>Windows</td>
<td>Total Profit</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td><strong>Batches Produced</strong></td>
<td></td>
<td>2</td>
<td>6</td>
<td></td>
<td></td>
<td>$36,000</td>
<td></td>
</tr>
</tbody>
</table>

Click on Spreadsheet to Open in Excel
### In Excel

#### Adjustable Cells

<table>
<thead>
<tr>
<th>Cell</th>
<th>Name</th>
<th>Final Value</th>
<th>Reduced Cost</th>
<th>Objective Coefficient</th>
<th>Allowable Increase</th>
<th>Allowable Decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$12</td>
<td>Batches Produced Doors</td>
<td>2</td>
<td>0</td>
<td>3,000</td>
<td>4,500</td>
<td>3,000</td>
</tr>
<tr>
<td>$D$12</td>
<td>Batches Produced Windows</td>
<td>6</td>
<td>0</td>
<td>5,000</td>
<td>1E+30</td>
<td>3,000</td>
</tr>
</tbody>
</table>

#### Constraints

<table>
<thead>
<tr>
<th>Cell</th>
<th>Name</th>
<th>Final Value</th>
<th>Shadow Price</th>
<th>Constraint R.H. Side</th>
<th>Allowable Increase</th>
<th>Allowable Decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$7</td>
<td>Plant 1 Used</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>1E+30</td>
<td>2</td>
</tr>
<tr>
<td>$E$8</td>
<td>Plant 2 Used</td>
<td>12</td>
<td>1,500</td>
<td>12</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>$E$9</td>
<td>Plant 3 Used</td>
<td>18</td>
<td>1,000</td>
<td>18</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>
Parametric Linear Programming

• **One parameter at a time**
  – Sensitivity analysis involves changing one parameter at a time in the original model to check its effect on the optimal solution.

• **Many parameters at a time**
  – By contrast, parametric linear programming (or parametric programming for short) involves the systematic study of how the optimal solution changes as many of the parameters change simultaneously over some range.
Simplex Algo: Implementation Considerations

- Number of ordinary functional constraints affects computation of Simplex the most
  - Computation time tends to be roughly proportional to the cube of this number, so that doubling this number may multiply the computation time by a factor of approximately 8.

- The number of variables
  - By contrast, the number of variables is a relatively minor factor. Thus, doubling the number of variables probably will not even double the computation time.

- Sparsity of constraint coefficients
  - A third factor of some importance is the density of the table of constraint coefficients (i.e., the proportion of the coefficients that are not zero), because this affects the computation time per iteration.

- Number of iterations is 2X functional constraints
  - One common rule of thumb for the number of iterations is that it tends to be roughly twice the number of functional constraints.
Solve LP using Interior Point Algorithm

- Radically different from the simplex method, Karmarkar’s algorithm an iterative algorithm
• Chapter 5
Some more Theory of Simplex Method

• General geometric and algebraic properties
  – Relationship between CPF solutions, and systems of eqns, and basic solutions and nonbasic vars
  – Number possible CPF solutions in a LP model (N+M choose M)
  – Going from CPF solution to CPF (sets of simulat

• The matrix form of the simplex method

• The fundamental insight (FI)
  – a property of the simplex method that enables us to deduce how changes that are made in the original model get carried along to the final simplex tableau
  – Cycle thru basis B generating a basic feasible solution and corresponding Z; To do this all we need is the initial Tableau and the current basis B (which updated for each iteration based upon entering and leaving variables)
  – This insight will provide the key to the important topics of duality theory and sensitivity analysis
3 Properties of CPF Solutions

• **Three key properties of CPF solutions that hold for any linear programming problem that has feasible solutions and a bounded feasible region.**

• **Property 1:**
  
  (a) If there is exactly one optimal solution, then it must be a CPF solution. (b) If there are multiple optimal solutions (and a bounded feasible region), then at least two must be adjacent CPF solutions.

• **Property 2:** There are only a finite number of CPF solutions

\[
\binom{m+n}{n} = \frac{(m+n)!}{m!n!}
\]

• **Property 3:** If a CPF solution has no adjacent CPF solutions that are better (as measured by Z), then there are no better CPF solutions anywhere.
CPF Solution: n decision vars $\rightarrow$ n constraint boundaries

- The constraint boundary equation for any constraint is obtained by replacing its $\leq$, or $\geq$ sign by an $=$ sign.
- For any linear programming problem with n decision variables, each CPF solution lies at the intersection of n constraint boundaries; i.e., it is the simultaneous solution of a system of n constraint boundary equations.

**TABLE 5.1** Defining equations for each CPF solution for the Wyndor Glass Co. problem

<table>
<thead>
<tr>
<th>CPF Solution</th>
<th>Defining Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>$x_1 = 0$</td>
</tr>
<tr>
<td>(0, 6)</td>
<td>$x_1 = 0$</td>
</tr>
<tr>
<td>(2, 6)</td>
<td>$2x_2 = 12$</td>
</tr>
<tr>
<td>(4, 3)</td>
<td>$3x_1 + 2x_2 = 18$</td>
</tr>
<tr>
<td>(4, 0)</td>
<td>$x_1 = 4$</td>
</tr>
<tr>
<td>(4, 0)</td>
<td>$x_1 = 4$</td>
</tr>
<tr>
<td>(6, 0)</td>
<td>$x_2 = 0$</td>
</tr>
<tr>
<td>(4, 0)</td>
<td>$x_2 = 0$</td>
</tr>
</tbody>
</table>
Defining Equations (CP Solution)

- Each corner-point solution is the simultaneous solution of a system of $n$ constraint boundary equations, which we called its defining equations.
\[ \binom{5}{3} = 10 \text{ possible solutions} \]
Three-variable LP problem

- Constraints
  - $x_1 \leq 4$
  - $x_2 \leq 4$
  - $x_1 + x_2 \leq 6$
  - $-x_1 + 2x_3 \leq 4$
  - $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$

- vertices:
  - $(0, 0, 2)$
  - $(2, 4, 3)$
  - $(0, 4, 2)$
  - $(0, 0, 0)$
  - $(2, 4, 0)$
  - $(4, 0, 0)$
  - $(4, 2, 4)$
  - $(4, 0, 4)$
From Geometry to Algebra

From intersection of constraint boundaries to simultaneous solution of constraint boundary equations

• When you shift from a geometric viewpoint to an algebraic one, intersection of constraint boundaries changes to simultaneous solution of constraint boundary equations.

• The $n$ (i.e., number of decision variables) constraint boundary equations yielding (defining) a CPF solution are its defining equations, where deleting one of these equations yields a line whose feasible segment is an edge of the feasible region.
Entering and leaving graphically speaking

• When the simplex method chooses an entering basic variable, the geometric interpretation is that it is choosing one of the edges emanating from the current CPF solution to move along.
• Increasing this variable from zero (and simultaneously changing the values of the other basic variables accordingly) corresponds to moving along this edge.
• Having one of the basic variables (the leaving basic variable) decrease so far that it reaches zero corresponds to reaching the first new constraint boundary at the other end of this edge of the feasible region.
Augmented Form

(1) \[ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} = b_1 \]

(2) \[ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + x_{n+2} = b_2 \]

\[ \vdots \]

(m) \[ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + x_{n+m} = b_m, \]
Defining Equations (CP Solution) and Indicator Variables (Basic Solution)

- Recall that each corner-point solution is the simultaneous solution of a system of \( n \) constraint boundary equations, which we called its defining equations.
- The key question is:
  - How do we tell whether a particular constraint boundary equation is one of the defining equations when the problem is in augmented form?
- Each constraint has an indicating variable that completely indicates (by whether its value is zero) whether that constraint’s boundary equation is satisfied by the current solution.
### Indicator Variables for Active Constraint Boundary

**TABLE 5.3 Indicating variables for constraint boundary equations**

<table>
<thead>
<tr>
<th>Type of Constraint</th>
<th>Form of Constraint</th>
<th>Constraint in Augmented Form</th>
<th>Constraint Boundary Equation</th>
<th>Indicating Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nonnegativity</td>
<td>$x_j \geq 0$</td>
<td>$x_j \geq 0$</td>
<td>$x_j = 0$</td>
<td>$x_j$</td>
</tr>
<tr>
<td>Functional ($\leq$)</td>
<td>$\sum_{j=1}^{n} a_j x_j \leq b_i$</td>
<td>$\sum_{j=1}^{n} a_j x_j + x_{n+1} = b_i$</td>
<td>$\sum_{j=1}^{n} a_j x_j = b_i$</td>
<td>$x_{n+1}$</td>
</tr>
<tr>
<td>Functional ($=$)</td>
<td>$\sum_{j=1}^{n} a_j x_j = b_i$</td>
<td>$\sum_{j=1}^{n} a_j x_j + \bar{x}_{n+1} = b_i$</td>
<td>$\sum_{j=1}^{n} a_j x_j = b_i$</td>
<td>$\bar{x}_{n+1}$</td>
</tr>
<tr>
<td>Functional ($\geq$)</td>
<td>$\sum_{j=1}^{n} a_j x_j \geq b_i$</td>
<td>$\sum_{j=1}^{n} a_j x_j + x_{n+1} - x_s = b_i$</td>
<td>$\sum_{j=1}^{n} a_j x_j = b_i$</td>
<td>$x_{n+1} - x_s$</td>
</tr>
</tbody>
</table>

*Indicating variable = 0 $\Rightarrow$ constraint boundary equation satisfied; indicating variable $\neq 0$ $\Rightarrow$ constraint boundary equation violated.
Each basic soln has m basic vars (> 0)

- Each basic solution has m basic variables, and the rest of the variables are nonbasic variables set equal to zero. (The number of nonbasic variables equals n plus the number of surplus variables if we have them)
- The values of the basic variables are given by the simultaneous solution of the system of m equations for the problem in augmented form (after the nonbasic variables are set to zero).
- This basic solution is the augmented corner-point solution whose n defining equations are those indicated by the nonbasic variables.
Indicating variables for the constraint boundary equations of the Wyndor Glass Co. problem*

Indicator variable $X_i = 0$ if corresponding equation is part of the simultaneous solution of $n$ defining equations (that make up the corner point)

Non-basic Variable (0 in the basic solution)

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Constraint in Augmented Form</th>
<th>Constraint Boundary Equation</th>
<th>Indicating Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1 \geq 0$</td>
<td>$x_1 \geq 0$</td>
<td>$x_1 = 0$</td>
<td>$x_1$</td>
</tr>
<tr>
<td>$x_2 \geq 0$</td>
<td>$x_2 \geq 0$</td>
<td>$x_2 = 0$</td>
<td>$x_2$</td>
</tr>
<tr>
<td>$x_1 \leq 4$</td>
<td>(1) $x_1 + x_3 = 4$</td>
<td>$x_1 = 4$</td>
<td>$x_3$</td>
</tr>
<tr>
<td>$2x_2 \leq 12$</td>
<td>(2) $2x_2 + x_4 = 12$</td>
<td>$2x_2 = 12$</td>
<td>$x_4$</td>
</tr>
<tr>
<td>$3x_1 + x_2 \leq 18$</td>
<td>(3) $3x_1 + 2x_2 + x_5 = 18$</td>
<td>$3x_1 + 2x_2 = 18$</td>
<td>$x_5$</td>
</tr>
</tbody>
</table>

*Indicating variable = 0 ⇒ constraint boundary equation satisfied; indicating variable ≠ 0 ⇒ constraint boundary equation violated.
Active Constraint Boundary

**Non-basic variable**

**i.e., x=0 in the Basic solution**

- Thus, whenever a constraint boundary equation is one of the defining equations for a corner-point solution, its indicating variable has a value of zero in the augmented form of the problem.
- Each such indicating variable is called a nonbasic variable for the corresponding basic solution.
• A BF solution is a basic solution where all $m$ basic variables are nonnegative ($\geq 0$).
• A BF solution is said to be degenerate if any of these $m$ variables equals zero.
### TABLE 5.5 BF solutions for the Wyndor Glass Co. problem

<table>
<thead>
<tr>
<th>CPF Solution</th>
<th>Defining Equations</th>
<th>BF Solution</th>
<th>Nonbasic Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>$x_1 = 0$</td>
<td>(0, 0, 4, 12, 18)</td>
<td>$x_1$</td>
</tr>
<tr>
<td></td>
<td>$x_2 = 0$</td>
<td></td>
<td>$x_2$</td>
</tr>
<tr>
<td>(0, 6)</td>
<td>$x_1 = 0$</td>
<td>(0, 6, 4, 0, 6)</td>
<td>$x_1$</td>
</tr>
<tr>
<td></td>
<td>$2x_2 = 12$</td>
<td></td>
<td>$x_4$</td>
</tr>
<tr>
<td>(2, 6)</td>
<td>$2x_2 = 12$</td>
<td>(2, 6, 2, 0, 0)</td>
<td>$x_4$</td>
</tr>
<tr>
<td></td>
<td>$3x_1 + 2x_2 = 18$</td>
<td></td>
<td>$x_5$</td>
</tr>
<tr>
<td>(4, 3)</td>
<td>$3x_1 + 2x_2 = 18$</td>
<td>(4, 3, 0, 6, 0)</td>
<td>$x_5$</td>
</tr>
<tr>
<td></td>
<td>$x_1 = 4$</td>
<td></td>
<td>$x_3$</td>
</tr>
<tr>
<td>(4, 0)</td>
<td>$x_1 = 4$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x_2 = 0$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Maximize $Z = 3x_1 + 3x_2$, subject to

- $x_1 \leq 4$
- $2x_2 \leq 12$
- $2x_1 + 3x_2 \leq 18$
- $x_1 \geq 0$, $x_2 \geq 0$

The feasible region is shown in the diagram.
Notice that in each case the nonbasic variables necessarily are the indicating variables for the defining equations.
## TABLE 5.6 Basic infeasible solutions for the Wyndor Glass Co. problem

<table>
<thead>
<tr>
<th>Corner-Point Infeasible Solution</th>
<th>Defining Equations</th>
<th>Basic Infeasible Solution</th>
<th>Nonbasic Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 9)</td>
<td>$x_1 = 0$</td>
<td>(0, 9, 4, -6, 0)</td>
<td>$x_1$, $x_5$</td>
</tr>
<tr>
<td></td>
<td>$3x_1 + 2x_2 = 18$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(4, 6)</td>
<td>$2x_2 = 12$</td>
<td>(4, 6, 0, 0, -6)</td>
<td>$x_4$, $x_3$</td>
</tr>
<tr>
<td></td>
<td>$x_1 = 4$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(6, 0)</td>
<td>$3x_1 + 2x_2 = 18$</td>
<td>(6, 0, -2, 12, 0)</td>
<td>$x_5$, $x_2$</td>
</tr>
<tr>
<td></td>
<td>$x_2 = 0$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Maximize $Z = 3x_1 + 5x_2$, subject to

- $x_1 \leq 4$
- $2x_2 \leq 12$
- $2x_1 + 3x_2 \leq 18$

and $x_1 \geq 0$, $x_2 \geq 0$
2 Non-basic solutions

- The other two sets of nonbasic variables, (1) $x_1$ and $x_3$ and (2) $x_2$ and $x_4$, do not yield a basic solution, because setting either pair of variables equal to zero leads to having no solution for the system of Eqs. (1) to (3) given in Table 5.4

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Constraint in Augmented Form</th>
<th>Constraint Boundary Equation</th>
<th>Indicating Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1 \geq 0$</td>
<td>$x_1 \geq 0$</td>
<td>$x_1 = 0$</td>
<td>$x_1$</td>
</tr>
<tr>
<td>$x_2 \geq 0$</td>
<td>$x_2 \geq 0$</td>
<td>$x_2 = 0$</td>
<td>$x_2$</td>
</tr>
<tr>
<td>$x_1 \leq 4$</td>
<td>$(1) \quad x_1 + x_3 = 4$</td>
<td>$x_1 = 4$</td>
<td>$x_3$</td>
</tr>
<tr>
<td>$2x_2 \leq 12$</td>
<td>$(2) \quad 2x_2 + x_4 = 12$</td>
<td>$2x_2 = 12$</td>
<td>$x_4$</td>
</tr>
<tr>
<td>$3x_1 + x_2 \leq 18$</td>
<td>$(3) \quad 3x_1 + 2x_2 + x_5 = 18$</td>
<td>$3x_1 + 2x_2 = 18$</td>
<td>$x_5$</td>
</tr>
</tbody>
</table>

*Indicating variable = 0 $\Rightarrow$ constraint boundary equation satisfied; indicating variable $\neq 0$ $\Rightarrow$ constraint boundary equation violated.
Simplex cycles thru adjacent BF Solutions

<table>
<thead>
<tr>
<th>Iteration</th>
<th>CPF Solution</th>
<th>Defining Equations</th>
<th>BF Solution</th>
<th>Nonbasic Variables</th>
<th>Functional Constraints in Augmented Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(0, 0)</td>
<td>( x_1 = 0 ) \n ( x_2 = 0 )</td>
<td>(0, 0, 4, 12, 18)</td>
<td>( x_1 = 0 ) \n ( x_2 = 0 )</td>
<td>( x_1 + x_3 = 4 ) \n ( 2x_2 + x_4 = 12 ) \n ( 3x_1 + 2x_2 + x_5 = 18 )</td>
</tr>
<tr>
<td>1</td>
<td>(0, 6)</td>
<td>( x_1 = 0 ) \n ( 2x_2 = 12 )</td>
<td>(0, 6, 4, 0, 6)</td>
<td>( x_1 = 0 ) \n ( x_4 = 0 )</td>
<td>( x_1 + x_3 = 4 ) \n ( 2x_2 + x_4 = 12 ) \n ( 3x_1 + 2x_2 + x_5 = 18 )</td>
</tr>
<tr>
<td>2</td>
<td>(2, 6)</td>
<td>( 2x_2 = 12 ) \n ( 3x_1 + 2x_2 = 18 )</td>
<td>(2, 6, 2, 0, 0)</td>
<td>( x_4 = 0 ) \n ( x_5 = 0 )</td>
<td>( x_1 + x_3 = 4 ) \n ( 2x_2 + x_4 = 12 ) \n ( 3x_1 + 2x_2 + x_5 = 18 )</td>
</tr>
</tbody>
</table>

[3rd Column] In the third column, note how each iteration results in deleting one constraint boundary (defining equation) and substituting a new one to obtain the new CPF solution.
Simplex cycles thru adjacent BF Solutions

[3rd Column] In the third column, note how each iteration results in deleting one constraint boundary (defining equation) and substituting a new one to obtain the new CPF solution.

[2nd LAST 5] Similarly, note in the fifth column how each iteration results in deleting one nonbasic variable and substituting a new one to obtain the new BF solution. Furthermore, the nonbasic variables being deleted and added are the indicating variables for the defining equations being deleted and added in the third column.

[LAST; 6] The last column displays the initial system of equations [excluding Eq. (0)] for the augmented form of the problem, with the current basic variables shown in bold type. In each case, note how setting the nonbasic variables equal to zero and then solving this system of equations for the basic variables must yield the same solution for \((x_1, x_2)\) as the corresponding pair of defining equations in the third column.
Algebraic Simplex Method - Introduction

To demonstrate the simplex method, consider the following linear programming model:

Maximize \( Z = 20a_1 + 10a_2 \)

subject to

\[ a_1 - a_2 \leq 1 \]
\[ 3a_1 + a_2 \leq 7 \]

and \( a_1 \geq 0, \ a_2 \geq 0. \)

This is the model for Leo Coco's problem presented in the demo, Graphical Method. That demo describes how to find the optimal solution graphically, as displayed on the right.

Thus the optimal solution is \( a_1 = 0, \ a_2 = 7, \) and \( Z = 70. \)

We will now describe how the simplex method (an algebraic procedure) obtains this solution algebraically.
Algebraic Simplex Method - Formulation

Maximize $Z = 20x_1 + 10x_2$

subject to

$x_1 - x_2 \leq 1$

$3x_1 + x_2 \leq 7$

and $x_1 \geq 0, x_2 \geq 0$.

The Simplex Formulation

To solve this model, the simplex method needs a system of equations instead of inequalities for the functional constraints. The demo, Interpretation of Slack Variables, describes how this system of equations is obtained by introducing nonnegative slack variables, $x_8$ and $x_4$. The resulting equivalent form of the model is

Maximize $Z$

subject to

$(0) \quad Z - 20x_1 - 10x_2 = 0$

$(1) \quad x_1 - x_2 + x_8 = 1$

$(2) \quad 3x_1 + x_2 + x_4 = 7$

and $x_1 \geq 0, x_2 \geq 0, x_8 \geq 0, x_4 \geq 0$.

The simplex method begins by focusing on equations (1) and (2) above.
Set NonBasic Vars to 0; BFS if each basic var is nonnegative

Algebraic Simplex Method - Initial Solution

(0) \( Z - 20x_1 - 10x_2 + 0x_3 + 0x_4 = 0 \)

(1) \( 1x_1 - 1x_2 + 1x_3 + 0x_4 = 1 \)

(2) \( 3x_1 + 1x_2 + 0x_3 + 1x_4 = 7 \)

The Initial Solution

Consider the initial system of equations exhibited above. Equations (1) and (2) include two more variables than equations. Therefore, two of the variables (the nonbasic variables) can be arbitrarily assigned a value of zero in order to obtain a specific solution (the basic solution) for the other two variables (the basic variables). This basic solution will be feasible if the value of each basic variable is nonnegative. The best of the basic feasible solutions is known to be an optimal solution, so the simplex method finds a sequence of better and better basic feasible solutions until it finds the best one.

To begin the simplex method, choose the slack variables to be the basic variables, so \( x_1 \) and \( x_2 \) are the nonbasic variables to set equal to zero. The values of \( x_3 \) and \( x_4 \) now can be obtained from the system of equations.

The resulting basic feasible solution is \( x_1 = 0, x_2 = 0, x_3 = 1, \) and \( x_4 = 7 \). Is this solution optimal?

Question
Optimal?
Which nonbasic var should we enter?

Algebraic Simplex Method - Checking Optimality

(0) \[ Z = 20x_1 - 10x_2 + 0x_3 + 0x_4 = 0 \]
(1) \[ 1x_1 - 1x_2 + 1x_3 + 0x_4 = 1 \]
(2) \[ 3x_1 + 1x_2 + 0x_3 + 1x_4 = 7 \]

Checking for Optimality
To test whether the solution \( x_1 = 0, x_2 = 0, x_3 = 1, \) and \( x_4 = 7 \) is optimal, we rewrite equation (0) as
\[ Z = 0 + 20x_1 + 10x_2 \]
Since both \( x_1 \) and \( x_2 \) have positive coefficients, \( Z \) can be increased by increasing either one of these variables. Therefore, the current basic feasible solution is not optimal, so we need to perform an iteration of the simplex method to obtain a better basic feasible solution.
This begins by choosing the entering basic variable (the nonbasic variable chosen to become a basic variable for the next basic feasible solution).
Select X1;
so which basic var should leave?

Algebraic Simplex Method - Entering Basic Variable

(0) \[ Z - 20x_1 - 10x_2 + 0x_3 + 0x_4 = 0 \]
(1) \[ 1x_1 - 1x_2 + 1x_3 + 0x_4 = 1 \]
(2) \[ 3x_1 + 1x_2 + 0x_3 + 1x_4 = 7 \]

Selecting an Entering Basic Variable
The entering basic variable is: \( x_1 \)

Why? Again rewrite equation (0) as \( Z = 0 + 20x_1 + 10x_2 \).
The value of the entering basic variable will be increased from 0. Since \( x_1 \) has the largest positive coefficient, increasing \( x_1 \) will increase \( Z \) at the fastest rate. So select \( x_1 \).

This selection rule tends to minimize the number of iterations needed to reach an optimal solution. You’ll see later that this particular problem is an exception where this rule does not minimize the number of iterations.
Minimum Ratio Test

Algebraic Simplex Method - Leaving Basic Variable

(0) \[ Z = 20x_1 - 10x_2 + 0x_3 + 0x_4 = 0 \]
(1) \[ x_1 - x_2 + x_3 + 0x_4 = 1 \]
(2) \[ 3x_1 + 0x_2 + 0x_3 + x_4 = 7 \]

Selecting a Leaving Basic Variable

The entering basic variable is: \( x_1 \)

The leaving basic variable is: \( x_3 \)

Why? Choose the basic variable that reaches zero first as the entering basic variable \( (x_1) \) is increased (watch \( x_1 \) increase).

Click on ORTutor to watch \( x_1 \) increase!

\( x_3 = 1 - x_1 \) \( \Rightarrow x_1=1 \) takes \( x_3 \) to 0
\( x_4 = 7 - 3x_1 \) \( \Rightarrow x_1=7/3 \) takes \( x_4 \) to 0
So choose the basic variable with the minimum ratio (why?), i.e., \( x_3 \)

X3 or x4 are zero
When the Eq 1 or Eq2 are satisfied.
They are involved in the basic feasible solution
What if we increase $X_1$ until $X_4$ is zero?

Algebraic Simplex Method - Leaving Basic Variable

(0) $Z = 20x_1 - 10x_2 + 0x_3 + 0x_4 = 0$
(1) $1x_1 - 1x_2 + 1x_3 + 0x_4 = 1$
(2) $3x_1 + 1x_2 + 0x_3 + 1x_4 = 7$

Selecting a Leaving Basic Variable

The entering basic variable is: $x_1$

The leaving basic variable is: $x_3$

Why? Choose the basic variable that reaches zero first as the entering basic variable ($x_1$) is increased.

$x_3 = 1 - x_1 \Rightarrow x_1 = 1$ takes $x_3$ to 0

$x_4 = 7 - 3x_1 \Rightarrow x_1 = 7/3$ takes $x_4$ to 0

So choose the basic variable with the minimum ratio (why?), i.e., $X_3$

$X_1=0$ means $x_1$ axis (constraint is binding)

$X_2=0$ means $x_2$ axis (constraint is binding)

$X_3=0$ means Eq1 constraint is binding

$X_4=0$ means Eq2 constraint is binding
Pivot; \( x_1 \) enters and \( x_3 \) leaves

Algebraic Simplex Method - Gaussian Elimination

(0) \[
\begin{array}{cccccc}
Z & - & 20x_1 & - & 10x_2 & + & 0x_3 & + & 0x_4 & = & 0 \\
(1) & 0 & 1 & 1 & x_1 & - & 1x_2 & + & 1x_3 & + & 0x_4 & = & 1 \\
(2) & 0 & 3 & x_1 & + & 1x_2 & + & 0x_3 & + & 1x_4 & = & 7 \\
\end{array}
\]

Scaling the Pivot Row

In order to determine the new basic feasible solution, we need to convert the system of equations into proper form from Gaussian elimination. The coefficient of the entering basic variable \( (x_1) \) in the equation of the leaving basic variable (equation (1)) must be 1.

The current value of this coefficient is: 1

Therefore, nothing needs to be done to this equation.
Example 1. Consider the system in canonical form:

\[
\begin{align*}
x_1 + x_4 + x_5 - x_6 &= 5 \\
x_2 + 2x_4 - 3x_5 + x_6 &= 3 \\
x_3 - x_4 + 2x_5 - x_6 &= -1.
\end{align*}
\]

Let us find the basic solution having basic variables \(x_4, x_5, x_6\). We set up the coefficient array below:

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 & -1 & 5 \\
0 & 1 & 0 & 2 & -3 & 1 & 3 \\
0 & 0 & 1 & -1 & 2 & -1 & -1
\end{bmatrix}
\]

The circle indicated is our first pivot element and corresponds to the replacement of \(x_1\) by \(x_4\) as a basic variable. After pivoting we obtain the array

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 & -1 & 5 \\
-2 & 1 & 0 & 0 & \circ & 3 & -7 \\
1 & 0 & 1 & 0 & 3 & -2 & 4
\end{bmatrix}
\]

and again we have circled the next pivot element indicating our intention to replace \(x_2\) by \(x_5\). We then obtain

\[
\begin{bmatrix}
3/5 & 1/5 & 0 & 1 & 0 & -2/5 & 18/5 \\
2/5 & -1/5 & 0 & 0 & 1 & -3/5 & 7/5 \\
-1/5 & 3/5 & 1 & 0 & 0 & -1/5 & -1/5
\end{bmatrix}
\]

Continuing, there results

\[
\begin{bmatrix}
1 & -1 & -2 & 1 & 0 & 0 & 4 \\
1 & -2 & -3 & 0 & 1 & 0 & 2 \\
1 & -3 & -5 & 0 & 0 & 1 & 1
\end{bmatrix}
\]

From this last canonical form we obtain the new basic solution

\[x_4 = 4, \quad x_5 = 2, \quad x_6 = 1.\]
Algebraic Simplex Method - Gaussian Elimination

\[(0) \quad Z = -20x_1 - 10x_2 + 0x_3 + 0x_4 = 0\]
\[(1) \quad 1x_1 - 1x_2 + 1x_3 + 0x_4 = 1\]
\[(2) \quad 3x_1 + 1x_2 + 0x_3 + 1x_4 = 7\]

Eliminating \(x_1\) from the Other Equations

Next, we need to obtain a coefficient of zero for the entering basic variable \((x_1)\) in every other equation (equations (0) and (2)).

The coefficient of \(x_1\) in equation (0) is: -20
To obtain a coefficient of 0 we need to:

Add 20 times equation (1) to equation (0).

The coefficient of \(x_1\) in equation (2) is: 3
Therefore, to obtain a coefficient of 0 we need to:

Subtract 3 times equation (1) from equation (2).
Algebraic Simplex Method - Checking Optimality

\[(0) \quad Z + 0x_1 - 30x_2 + 20x_3 + 0x_4 = 20\]
\[(1) \quad 1x_1 - 1x_2 + 1x_3 + 0x_4 = 1\]
\[(2) \quad 0x_1 + 4x_2 - 3x_3 + 1x_4 = 4\]

Checking for Optimality
The new basic feasible solution is \(x_1 = 1, x_2 = 0, x_3 = 0,\) and \(x_4 = 4,\) which yields \(Z = 20.\)
This ends iteration 1.

Is the current solution optimal? No.
Why? Rewrite equation (0) as \(Z = 20 + 30x_2 - 20x_3.\)
Since \(x_2\) has a positive coefficient, increasing \(x_2\) from zero will increase \(Z.\) So the current basic feasible solution is not optimal.
Is the current BFS optimal?

Algebraic Simplex Method - Checking Optimality

\[(0) \quad Z + 0x_1 - 30x_2 + 20x_3 + 0x_4 = 20\]
\[(1) \quad 1x_1 - 1x_2 + 1x_3 + 0x_4 = 1\]
\[(2) \quad 0x_1 + 4x_2 - 3x_3 + 1x_4 = 4\]

Checking for Optimality

The new basic feasible solution is \(x_1 = 1, x_2 = 0, x_3 = 0,\) and \(x_4 = 4,\) which yields \(Z = 20.\)

This ends iteration 1.

Is the current solution optimal? No.

Why? Rewrite equation (0) as \(Z = 20 + 30x_2 - 20x_3.\)

Since \(x_2\) has a positive coefficient, increasing \(x_2\) from zero will increase \(Z.\) So the current basic feasible solution is not optimal.
Choose X2 to enter

Algebraic Simplex Method - Entering Basic Variable

(0) \[ Z + 0x_1 - 30x_2 + 20x_3 + 0x_4 = 20 \]
(1) \[ 1x_1 - 1x_2 + 1x_3 + 0x_4 = 1 \]
(2) \[ 0x_1 + 4x_2 - 3x_3 + 1x_4 = 4 \]

Selecting an Entering Basic Variable

The entering basic variable is: \( x_2 \)

Why? Again rewrite equation (0) as \( Z = 20 + 30x_2 - 20x_3 \).

The value of the entering basic variable will be increased from 0. Since \( x_2 \) has the largest (and only) positive coefficient, increasing \( x_2 \) will increase \( Z \) at the fastest rate. So select \( x_2 \).
X4 leaves since x2 can't move

Algebraic Simplex Method - Leaving Basic Variable

$(0) \quad Z + 0x_1 - 30x_2 + 20x_3 + 0x_4 = 20$

$(1) \quad 1x_1 - 1x_2 + 1x_3 + 0x_4 = 1$

$(2) \quad 0x_1 + 4x_2 - 3x_3 + 1x_4 = 4$

Selecting a Leaving Basic Variable

The entering basic variable is: $x_2$

The leaving basic variable is: $x_4$

Why? Choose the basic variable that reaches zero first as the entering basic variable ($x_2$) is increased (watch $x_2$ increase).

$x_4 = 0$ when $x_2 = 1$.

Since the coefficient of $x_2$ is negative in equation (1), $x_1$ will never reach zero, no matter how far $x_2$ is increased. Therefore, $x_4$ is the leaving basic variable.

X1 = 1 + x2 ➔ since x2 is pos x1 will never reach 0
X4 = 4 – 4x2 ➔ x2=1; so choose x4 to leave as it can reach zero
Watch X2 increase

Algebraic Simplex Method - Leaving Basic Variable

(0) \( Z + 0x_1 - 30x_2 + 20x_3 + 0x_4 = 20 \)

(1) \( \boxed{1x_1} - \ 1x_2 + \ 1x_3 + 0x_4 = 1 \)

(2) \( 0x_1 + 4x_2 - 3x_3 + \boxed{1x_4} = 4 \)

Selecting a Leaving Basic Variable

The entering basic variable is: \( x_2 \)

The leaving basic variable is: \( x_4 \)

Why? Choose the basic variable that reaches zero first as the entering basic variable \( (x_2) \) is increased (watch \( x_2 \) increase).

\[ x_4 = 0 \text{ when } x_2 = 1. \]

Since the coefficient of \( x_2 \) is negative in equation \( (1) \), \( x_1 \) will never reach zero, no matter how far \( x_2 \) is increased. Therefore, \( x_4 \) is the leaving basic variable.
Algebraic Simplex Method - Leaving Basic Variable

(0) \[ Z + 0x_1 - 30x_2 + 20x_3 + 0x_4 = 20 \]

(1) \[
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 4 & 0 & 3 & 1
\end{bmatrix}
\]
\[ x_3 = 1 \]

(2) \[
\begin{bmatrix}
0 & 1 & 3 & 0 & 1
\end{bmatrix}
\]
\[ x_4 = 4 \]

Selecting a Leaving Basic Variable

The entering basic variable is: \( x_2 \)

The leaving basic variable is: \( x_4 \)

Why? Choose the basic variable that reaches zero first as the entering basic variable (\( x_2 \)) is increased. (watch \( x_2 \) increase).

\[ x_4 = 0 \text{ when } x_2 = 1. \]

Since the coefficient of \( x_2 \) is negative in equation (1), \( x_1 \) will never reach zero, no matter how far \( x_2 \) is increased. Therefore, \( x_4 \) is the leaving basic variable.
Simplex in Matrix Form

Maximize \( Z = cx \),
subject to
\( Ax \leq b \) and \( x \geq 0 \),

where \( c \) is the row vector
\( c = [c_1, c_2, \ldots, c_n] \),

\( x, b, \) and \( 0 \) are the column vectors such that
\[
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}, \quad
\begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_m
\end{bmatrix}, \quad
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix},
\]

and \( A \) is the matrix
\[
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}.
\]

- Streamlined version of the original simplex procedure
- Moving from basis to basis (in terms of the basic variables)
Augmented Form in Matrix Form

To obtain the augmented form of the problem, introduce the column vector of slack variables

\[
\mathbf{x}_s = \begin{bmatrix}
  x_{n+1} \\
  x_{n+2} \\
  \vdots \\
  x_{n+m}
\end{bmatrix}
\]

so that the constraints become

\[
\begin{bmatrix} \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \mathbf{b} \quad \text{and} \quad \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} \geq 0,
\]

where \( \mathbf{I} \) is the \( m \times m \) identity matrix, and the null vector \( \mathbf{0} \) now has \( n + m \) elements. (We comment at the end of the section about how to deal with problems that are not in our standard form.)
$X_B$ are the basic variables (>0)

- Given basic and nonbasic variables the resulting basic solution is the solution to the $m$ equations

$$[A, I] \begin{bmatrix} X \\ X_s \end{bmatrix} = b,$$

in which the $n$ nonbasic variables from the $n + m$ elements of

$$\begin{bmatrix} X \\ X_s \end{bmatrix}$$

are set equal to zero. Eliminating these $n$ variables by equating them to zero leaves a set of $m$ equations in $m$ unknowns (the basic variables). This set of equations can be denoted by

$$Bx_B = b,$$
B the Basis Matrix $\rightarrow$ Solve to get basic solution

- The simplex method introduces only basic variables such that $B$ is nonsingular, so that $B^{-1}$ always will exist (inverse).

\[
B = \begin{bmatrix}
B_{11} & B_{12} & \cdots & B_{1m} \\
B_{21} & B_{22} & \cdots & B_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
B_{m1} & B_{m2} & \cdots & B_{mm}
\end{bmatrix}
\]

\[
B^{-1}Bx_B = B^{-1}b.
\]

Since $B^{-1}B = I$, the desired solution for the basic variables is

\[
x_B = B^{-1}b.
\]
Get Basic Feasible Solution

Since $B^{-1}B = I$, the desired solution for the basic variables is

$$x_B = B^{-1}b.$$  

Let $c_B$ be the vector whose elements are the objective function coefficients (including zeros for slack variables) for the corresponding elements of $x_B$. The value of the objective function for this basic solution is then

$$Z = c_Bx_B = c_BB^{-1}b.$$  

WynDor Problem

$$c = [3, 5], \quad [A, I] = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x_s = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$
Drop non-basic variables $x_1$ and $x_2$ from $[A,I]$ to yield the Basic matrix

$$c = [3, 5], \quad [A, I] = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x_5 = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$ 

**Iteration 0**

1. Get BF Solution

$$x_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = B^{-1}, \quad \text{so} \quad \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix},$$

$$c_B = [0, 0, 0], \quad \text{so} \quad Z = [0, 0, 0] \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = 0.$$

**Optimal value**

$$Z = c_B x_B = c_B B^{-1} b.$$
Drop non-basic variables $x_1$ and $x_2$ from $[A, I]$ to yield the Basic matrix.

\[
c = [3, 5], \quad [A, I] = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x_5 = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}.
\]

**Iteration 0**

1. Get BF Solution

\[
x_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = B^{-1}, \quad \text{so} \quad \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix},
\]

\[
c_B = [0, 0, 0], \quad \text{so} \quad Z = [0, 0, 0] \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = 0.
\]

Notice correspondence with Shadow Prices; a unit of increase in $b$ contributes $c_B B^{-1}$ to the objective function; $c_B B^{-1}$ becomes our shadow price.
Optimal value

Iteration 0

\[
\begin{align*}
&c = [3, 5], \quad [A, I] = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x_s = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} \\
&B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = B^{-1}, \quad \text{so} \quad \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix},
\end{align*}
\]

\[
c_B = [0, 0, 0], \quad \text{so} \quad Z = [0, 0, 0] \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = 0.
\]

Iteration 1

\[
\begin{align*}
&B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix},
\end{align*}
\]

so

\[
\begin{align*}
\begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix},
\end{align*}
\]

\[
c_B = [0, 5, 0], \quad \text{so} \quad Z = [0, 5, 0] \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix} = 30.
\]
\[ c = [3, 5], \quad [A, I] = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}, \quad x = [x_1, x_2], \quad x_s = [x_3, x_4, x_5] \]

**Iteration 0**

\[ x_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = B^{-1}, \quad \text{so} \quad \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} \]

\[ c_B = [0, 0, 0], \quad \text{so} \quad Z = [0, 0, 0] \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = 0. \]

**Iteration 1**

\[ x_B = \begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad B^{-1} = \]

\[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \]

\[ \begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix}, \quad c_B = [0, 5, 0], \quad \text{so} \quad Z = [0, 5, 0] \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix} = 30. \]

**Iteration 2**

\[ x_B = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix}, \quad B^{-1} = \]

\[ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 1/2 & 0 \\ 0 & -1/3 & 1/3 \end{bmatrix} \]

\[ \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 1/2 & 0 \\ 0 & -1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}, \quad x_B = B^{-1}b. \]

\[ Z = c_B x_B = c_B B^{-1}b. \]

\[ c_B = [0, 5, 3], \quad \text{so} \quad Z = [0, 5, 3] \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} = 36. \]
Iteration 0

\( c = [3, 5], \quad [A, I] = \begin{bmatrix}
1 & 0 & 1 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 \\
3 & 2 & 0 & 0 & 1
\end{bmatrix}, \quad b = \begin{bmatrix}
4 \\
12 \\
18
\end{bmatrix}, \quad x = \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}, \quad x_s = \begin{bmatrix}
x_3 \\
x_4 \\
x_5
\end{bmatrix}

\[
x_B = \begin{bmatrix}
x_3 \\
x_4 \\
x_5
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} = B^{-1}, \quad \text{so} \quad \begin{bmatrix}
x_3 \\
x_4 \\
x_5
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
4 \\
12 \\
18
\end{bmatrix} = \begin{bmatrix}
4 \\
12 \\
18
\end{bmatrix}
\]

\( c_B = [0, 0, 0], \quad \text{so} \quad Z = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
4 \\
12 \\
18
\end{bmatrix} = 0. \)

Iteration 1

\[
x_B = \begin{bmatrix}
x_3 \\
x_2 \\
x_5
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & -1
\end{bmatrix}, \quad B^{-1} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_3 \\
x_2 \\
x_5
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix} \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\( c_B = [0, 5, 0], \quad \text{so} \quad Z = \begin{bmatrix}
0 \\
5 \\
0
\end{bmatrix} \begin{bmatrix}
4 \\
6 \\
6
\end{bmatrix} = 30. \)

Iteration 2

\[
x_B = \begin{bmatrix}
x_3 \\
x_2 \\
x_1
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 & 1 \\
0 & 2 & 0 \\
0 & 2 & 3
\end{bmatrix}, \quad B^{-1} = \begin{bmatrix}
1 & \frac{1}{3} & -\frac{1}{3} \\
0 & \frac{1}{2} & 0 \\
0 & -\frac{1}{3} & \frac{1}{3}
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_3 \\
x_2 \\
x_1
\end{bmatrix} = \begin{bmatrix}
1 & \frac{1}{3} & -\frac{1}{3} \\
0 & \frac{1}{2} & 0 \\
0 & -\frac{1}{3} & \frac{1}{3}
\end{bmatrix} \begin{bmatrix}
4 \\
12 \\
18
\end{bmatrix} = \begin{bmatrix}
2 \\
6 \\
2
\end{bmatrix}
\]

\( x_B = B^{-1}b. \)

\[
Z = c_Bx_B = c_BB^{-1}b
\]

\[c_B = [0, 5, 3], \quad \text{so} \quad Z = \begin{bmatrix}
0 \\
5 \\
3
\end{bmatrix} \begin{bmatrix}
2 \\
6 \\
2
\end{bmatrix} = 36. \]
Put it all together in Tableau and Matrix Form

Initial Tableau Matrix

\[
\begin{bmatrix}
1 & -c & 0 \\
0 & A & I
\end{bmatrix}
\begin{bmatrix}
Z \\
x \\
x_s
\end{bmatrix}
=
\begin{bmatrix}
0 \\
b
\end{bmatrix}.
\]

Notice correspondence with Shadow Prices; a unit of increase in \( b \) contributes \( c_B B^{-1} \) to the objective function; \( c_B B^{-1} \) becomes our shadow price.

TABLE 5.8 Initial and later simplex tableaux in matrix form

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basic Variable</th>
<th>Eq.</th>
<th>Coefficient of:</th>
<th>Right Side</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>( Z )</td>
<td>Original Variables</td>
</tr>
<tr>
<td>0</td>
<td>( Z )</td>
<td>(0)</td>
<td>1</td>
<td>(-c)</td>
</tr>
<tr>
<td></td>
<td>( x_B )</td>
<td>(1, 2, ... , m)</td>
<td>0</td>
<td>( A )</td>
</tr>
</tbody>
</table>

Any

<table>
<thead>
<tr>
<th>( Z )</th>
<th>( x_B )</th>
<th>(0)</th>
<th>(1, 2, ... , m)</th>
<th>1</th>
<th>( c_B B^{-1} A - c )</th>
<th>( c_B B^{-1} A )</th>
<th>( c_B B^{-1} b )</th>
<th>( c_B B^{-1} b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z )</td>
<td>( x_B )</td>
<td>(0)</td>
<td>(1, 2, ... , m)</td>
<td>1</td>
<td>( B^{-1} A )</td>
<td>( B^{-1} A )</td>
<td>( B^{-1} b )</td>
<td>( B^{-1} b )</td>
</tr>
</tbody>
</table>
Each Iteration: use $B^{-1}$

In particular, after any iteration, $x_B = B^{-1}b$ and $Z = c_BB^{-1}b$, so the right-hand sides of the new set of equations have become

$$\begin{bmatrix} Z \\ x_B \end{bmatrix} = \begin{bmatrix} 1 & c_BB^{-1} \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{bmatrix} c_BB^{-1}b \\ B^{-1}b \end{bmatrix}.$$}

Because we perform the same series of algebraic operations on both sides of the original set of equations, we use this same matrix that premultiplies the original right-hand side to premultiply the original left-hand side. Consequently, since

$$\begin{bmatrix} 1 & c_BB^{-1} \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} 1 & -c & 0 \\ 0 & A & I \end{bmatrix} = \begin{bmatrix} 1 & c_BB^{-1}A - c & c_BB^{-1} \\ 0 & B^{-1}A & B^{-1} \end{bmatrix},$$

we have

$$\begin{bmatrix} 1 & c_BB^{-1}A - c & c_BB^{-1} \\ 0 & B^{-1}A & B^{-1} \end{bmatrix} \begin{bmatrix} Z \\ x \\ x_s \end{bmatrix} = \begin{bmatrix} c_BB^{-1}b \\ B^{-1}b \end{bmatrix}.$$
Example. To illustrate this matrix form for the current set of equations, we will show how it yields the final set of equations resulting from iteration 2 for the Wyndor Glass Co. problem. Using the $B^{-1}$ and $c_B$ given for iteration 2 at the end of the preceding subsection, we have

$$B^{-1}A = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$c_BB^{-1} = [0, 5, 3] \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} = [0, \frac{3}{2}, 1],$$

$$c_BB^{-1}A - c = [0, 5, 3] \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} - [3, 5] = [0, 0].$$

Also, by using the values of $x_B = B^{-1}b$ and $Z = c_BB^{-1}b$ calculated at the end of the preceding subsection, these results give the following set of equations:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{3}{2} & 1 \\ 0 & 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} Z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 36 \\ 2 \\ 6 \\ 2 \\ 0 \\ 0 \end{bmatrix},$$

as shown in the final simplex tableau in Table 4.8.
Summary of the Revised Simplex Method.

1. **Initialization:** Introduce slack variables, etc., to obtain the initial basic variables, as described in Chap. 4. This yields the initial $x_B$, $c_B$, $B$, and $B^{-1}$ (where $B = I = B^{-1}$ under our current assumption that the problem being solved fits our standard form). Then go to the optimality test.

2. **Iteration:**
   - **Step 1.** Determine the entering basic variable: Refer to the coefficients of the nonbasic variables in Eq. (0) that were obtained in the preceding application of the optimality test below. Then (just as described in Sec. 4.4), select the variable with the negative coefficient having the largest absolute value as the entering basic variable.
   - **Step 2.** Determine the leaving basic variable: Use the matrix expressions, $B^{-1}A$ (for the coefficients of the original variables) and $B^{-1}$ (for the coefficients of the slack variables), to calculate the coefficients of the entering basic variable in every equation except Eq. (0). Also use the preceding calculation of $x_B = B^{-1}b$ (see Step 3) to identify the right-hand sides of these equations. Then (just as described in Sec. 4.4), use the minimum ratio test to select the leaving basic variable.
   - **Step 3.** Determine the new BF solution: Update the basis matrix $B$ by replacing the column for the leaving basic variable by the corresponding column in $[A, I]$ for the entering basic variable. Also make the corresponding replacements in $x_B$ and $c_B$. Then derive $B^{-1}$ (as illustrated in Appendix 4) and set $x_B = B^{-1}b$.

3. **Optimality test:** Use the matrix expressions, $c_B B^{-1}A - c$ (for the coefficients of the original variables) and $c_B B^{-1}$ (for the coefficients of the slack variables), to calculate the coefficients of the nonbasic variables in Eq. (0). The current BF solution is optimal if and only if all of these coefficients are nonnegative. If it is optimal, stop. Otherwise, go to an iteration to obtain the next BF solution.
Entering and Leaving variables in Matrix From

• **Optimality test**
  – Check the coefficients of the nonbasic variables (if negative then $X_B$ is not optimal

• **Select Entering Value via matrix-based simplex**
  – Use the coefficients of the nonbasic variables
  – Select the variable with the largest absolute negative coefficient

• **Select Leaving Value via matrix-based simplex**
  – Minimum ratio test

• **Doing it in matrices just focus on the relevant portions, i.e., the column of the entering variable**
Put it all together in Tableau and Matrix Form

Initial Tableau Matrix

\[
\begin{bmatrix}
1 & -c & 0 \\
0 & A & I
\end{bmatrix}
\begin{bmatrix}
Z \\
x \\
x_s
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
b
\end{bmatrix}
\]

\[
[A, I] = 
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 \\
3 & 2 & 0 & 0 & 1
\end{bmatrix}
\]

\[
x_B = B^{-1} b.
\]

\[
Z = c_B x_B = c_B B^{-1} b.
\]

Notice correspondence with Shadow Prices; a unit of increase in \( b \) contributes \( c_B B^{-1} \) to the objective function; \( c_B B^{-1} \) becomes our shadow price

**TABLE 5.8 Initial and later simplex tableaux in matrix form**

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basic Variable</th>
<th>Eq.</th>
<th>Coefficient of:</th>
<th>Right Side</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( Z )</td>
<td>(0)</td>
<td>( Z )</td>
<td>Original Variables</td>
</tr>
<tr>
<td></td>
<td>( x_B )</td>
<td>(1, 2, \ldots, m)</td>
<td>( -c )</td>
<td>0</td>
</tr>
<tr>
<td>Any</td>
<td>( Z )</td>
<td>(0)</td>
<td>( Z )</td>
<td>Original Variables</td>
</tr>
<tr>
<td></td>
<td>( x_B )</td>
<td>(1, 2, \ldots, m)</td>
<td>( c_B B^{-1} A - c )</td>
<td>0</td>
</tr>
</tbody>
</table>
Revised Simplex leads to reduced complexity

- Each iteration need to calculate the inverse $B^{-1}$
- This can then be used to calculate all the numbers in the simplex tableau from the original parameters $(A, b, cB)$ of the problem. (This implication is the essence of the fundamental insight)
- Isolate calculations
  - Any one of these numbers can be obtained individually, usually by performing only a vector multiplication (one row times one column) instead of a complete matrix multiplication.
  - Therefore, the required numbers to perform an iteration of the simplex method can be obtained as needed without expending the computational effort to obtain all the numbers.
Fundamental Insight [a la HL]

Verbal description of fundamental insight: After any iteration, the coefficients of the slack variables in each equation immediately reveal how that equation has been obtained from the initial equations.

As one example of the importance of this insight, recall from Table 5.8 that the matrix formula for the optimal solution obtained by the simplex method is

$$x_B = B^{-1}b,$$

where $x_B$ is the vector of basic variables, $B^{-1}$ is the matrix of coefficients of slack variables for rows 1 to $m$ of the final tableau, and $b$ is the vector of original right-hand sides (resource availabilities). (We soon will denote this particular $B^{-1}$ by $S^*$..) Postoptimality analysis normally includes an investigation of possible changes in $b$. By using this for-
**Fl: Given \( B_i^{-1} \) and \([-c,0,0]\) and \([A,I,b]\)**

The coefficients of the slack variables in the current simplex tableau become \( c_B B^{-1} \) for row 0
And \( B^{-1} \) for the rest of the rows, where \( B \) is the current basis matrix

### TABLE 5.8 Initial and later simplex tableaux in matrix form

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basic Variable</th>
<th>Eq.</th>
<th>Coefficient of:</th>
<th>Right Side</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( Z ) ( x_B )</td>
<td>(0) (1, 2, \ldots, m)</td>
<td>( Z ) ( c ) ( A ) 0</td>
<td>0 ( b )</td>
</tr>
<tr>
<td>Any</td>
<td>( Z ) ( x_B )</td>
<td>(0) (1, 2, \ldots, m)</td>
<td>( c_B B^{-1} A - c ) ( B^{-1} A ) ( c_B B^{-1} b ) ( B^{-1} b )</td>
<td></td>
</tr>
</tbody>
</table>

Row 0 = \([-c, 0, 0]\) + \( c_B B^{-1} [A, I, b] \)
Rows 1 to \( m \) = \( B^{-1} [A, I, b] \)

Solution for the basic vars; while nonbasic variables are set to zero (0)
Fundamental Insight

- Fundamental Insight: Given $B_i^{-1}$ (inverse of the Basis matrix and and the initial tableau [-c,0,0 ] and [A, I, b] we can compute everything else in the current tableau (for every iteration).
Fundamental Insight [a la HL]

Verbal description of fundamental insight: After any iteration, the coefficients of the slack variables in each equation immediately reveal how that equation has been obtained from the initial equations.

As one example of the importance of this insight, recall from Table 5.8 that the matrix formula for the optimal solution obtained by the simplex method is

$$x_B = B^{-1}b,$$

where $x_B$ is the vector of basic variables, $B^{-1}$ is the matrix of coefficients of slack variables for rows 1 to $m$ of the final tableau, and $b$ is the vector of original right-hand sides (resource availabilities). (We soon will denote this particular $B^{-1}$ by $S^*$. ) Postoptimality analysis normally includes an investigation of possible changes in $b$. By using this for-
**TABLE 5.10** General notation for initial and final simplex tableaux in matrix form, illustrated by the Wyndor Glass Co. problem

### Initial Tableau

| Row 0: \( t = [-3, -5, 0, 0, 0, 0] = [-c, 0, 0] \) | \[
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 4 \\
0 & 2 & 0 & 1 & 0 & 12 \\
3 & 2 & 0 & 0 & 1 & 18
\end{bmatrix}
\] = \([A, I, b]\). |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Other rows:</td>
<td>(T = [1, 0, 0, 0, 4] = [A, I, b]).</td>
</tr>
</tbody>
</table>
| Combined: | \[
\begin{bmatrix}
t \\
T
\end{bmatrix} = \begin{bmatrix}
-c \\
0 \\
0
\end{bmatrix}.
\] |

### Final Tableau

| Row 0: \( t^* = [0, 0, \frac{3}{2}, 1, 36] = [z^* - c, y^*, z^*] \) | \[
\begin{bmatrix}
0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & 2 \\
0 & 1 & 0 & \frac{1}{2} & 0 & 6 \\
1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 2
\end{bmatrix}
\] = \([A^*, S^*, b^*]\). |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Other rows:</td>
<td>(T^* = [1, 0, 0, 0, 4] = [A^<em>, S^</em>, b^*]).</td>
</tr>
</tbody>
</table>
| Combined: | \[
\begin{bmatrix}
t^* \\
T^*
\end{bmatrix} = \begin{bmatrix}
z^* - c \\
y^* \\
z^*
\end{bmatrix}.
\] |

Row 0 = \([-c, 0, 0] + c_B B^{-1}[A, I, b]\)
Rows 1 to \(m = B^{-1}[A, I, b]\)
Fundamental Insight: Example on OR Tutor

The coefficients of the slack variables in the current simplex tableau become $c_B B^{-1}$ for row 0 and $B^{-1}$ for the rest of the rows, where $B$ is the current basis matrix.

<table>
<thead>
<tr>
<th>Initial Tableau:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Basic Variable</strong></td>
</tr>
<tr>
<td>$Z$</td>
</tr>
<tr>
<td>$x_3$</td>
</tr>
<tr>
<td>$x_4$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Final Tableau:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Coefficient of</strong></td>
</tr>
<tr>
<td>$Z$</td>
</tr>
<tr>
<td>$-c$</td>
</tr>
<tr>
<td>$A$</td>
</tr>
</tbody>
</table>

The fundamental insight is that, given just the initial tableau $(A, b, c)$ and the coefficients of the slack variables in the final tableau ($y^*$ and $S^*$), the rest of the final tableau can be calculated directly.
Fundamental Insight

Fundamental Insight - Introduction

To illustrate the fundamental insight of Section 5.3, consider the demonstration example for the simplex method shown below.

**Standard notation:**

Maximize \( Z = 20x_1 + 10x_2 \)

subject to \( x_1 - x_2 \leq 1 \)

\( 3x_1 + x_2 \leq 7 \)

and \( x_1 \geq 0, \ x_2 \geq 0. \)

**Matrix notation:**

Maximize \( Z = cx \) \quad c = [10 \ 20] \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \)

subject to \( Ax \) \quad A = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 17 \\ 0 \end{bmatrix} \)

and \( x \geq 0. \)

After introducing slack variables \( (s_3, s_4, \ldots) \), etc., the initial simplex tableau is

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Equation Number</th>
<th>Coefficient of $Z$</th>
<th>Coefficient of $x_1$</th>
<th>Coefficient of $x_2$</th>
<th>Coefficient of $x_3$</th>
<th>Coefficient of $x_4$</th>
<th>Right side</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>0</td>
<td>1</td>
<td>-20</td>
<td>-10</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$s_3$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$s_4$</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>7</td>
</tr>
</tbody>
</table>

\[
\begin{array}{cccc|c}
\text{Coefficient of} & x_1 & x_2 & x_3 & x_4 & \text{Right side} \\
\hline
-c & 0 & 0 & 0 & 0 & 0 \\
A & I & b
\end{array}
\]
Fundamental Insight - Initial and Final Tableaux

Initial Tableau:

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Equation Number</th>
<th>Coefficient of</th>
<th>Right side</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>0</td>
<td>$1$ $-20$ $-10$ $0$ $0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$x_8$</td>
<td>1</td>
<td>$0$ $1$ $-1$ $1$ $0$</td>
<td>$1$</td>
</tr>
<tr>
<td>$x_4$</td>
<td>2</td>
<td>$0$ $3$ $1$ $0$ $1$</td>
<td>$7$</td>
</tr>
</tbody>
</table>

Final Tableau:

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Equation Number</th>
<th>Coefficient of</th>
<th>Right side</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>0</td>
<td>$1$ $10$ $0$ $0$ $10$</td>
<td>$70$</td>
</tr>
<tr>
<td>$x_8$</td>
<td>1</td>
<td>$0$ $4$ $1$ $1$ $1$</td>
<td>$8$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>2</td>
<td>$0$ $3$ $1$ $0$ $1$</td>
<td>$7$</td>
</tr>
</tbody>
</table>

After performing three iterations of the simplex method to obtain an optimal solution, the final simplex tableau is shown above.
Fundamental Insight - Calculating Tableaux

**Initial Tableau:**

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Equation Number</th>
<th>Coefficient of $Z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>Right side</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>0</td>
<td>1</td>
<td>-20</td>
<td>-10</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_3$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$x_4$</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>7</td>
</tr>
</tbody>
</table>

$\begin{array}{c}\begin{array}{c}\text{Coefficient of} \\
\text{Right side}\end{array} \\
\hline x_1 \quad x_2 \quad x_3 \quad x_4 \quad \text{Right side} \\
\hline -c \quad 0 \quad 0 \quad 0 \\
A \quad I \quad b \end{array}$

**Final Tableau:**

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Equation Number</th>
<th>Coefficient of $Z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>Right side</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>0</td>
<td>1</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>70</td>
</tr>
<tr>
<td>$x_3$</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>$x_2$</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>7</td>
</tr>
</tbody>
</table>

$\begin{array}{c}\begin{array}{c}\text{Coefficient of} \\
\text{Right side}\end{array} \\
\hline x_1 \quad x_2 \quad x_3 \quad x_4 \quad \text{Right side} \\
\hline z^* - c \quad y^* \quad Z^* \\
A^* \quad S^* \quad b^* \end{array}$

The fundamental insight is that, given just the initial tableau $(A, b, c)$ and the coefficients of the slack variables in the final tableau $(y^* \text{ and } S^*)$, the rest of the final tableau can be calculated directly.
Fundamental Insight - Calculating Tableaux

Initial Tableau:

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Equation Number</th>
<th>Coefficient of</th>
<th>( Z )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>Right side</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z )</td>
<td>0</td>
<td></td>
<td>1</td>
<td>20</td>
<td>-10</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>1</td>
<td></td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>2</td>
<td></td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>7</td>
</tr>
</tbody>
</table>

Final Tableau:

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Equation Number</th>
<th>Coefficient of</th>
<th>( Z )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>Right side</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z )</td>
<td>0</td>
<td></td>
<td>1</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td>70</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>1</td>
<td></td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>2</td>
<td></td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>7</td>
</tr>
</tbody>
</table>

In particular, the rest of row 0 can be calculated directly as follows:

\[
z^* - c = y^* A - c = [0 \quad 10] \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} - [20 \quad 10] = [30 \quad 10] - [20 \quad 10] = [10 \quad 0].
\]

\[
Z^* = y^* b = [0 \quad 10] \begin{bmatrix} 1 \\ 7 \end{bmatrix} = [70], \text{ so row 0 is } \begin{bmatrix} 10 \quad 0 \quad 0 \quad 10 \quad 70 \end{bmatrix}.
\]
Fundamental Insight - Calculating Tableaux

Initial Tableau:

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Equation Number</th>
<th>Coefficient of</th>
<th>Right side</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$Z$</td>
<td>$\omega_1$</td>
</tr>
<tr>
<td>$Z$</td>
<td>0</td>
<td>1</td>
<td>-20</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>2</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

Final Tableau:

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Equation Number</th>
<th>Coefficient of</th>
<th>Right side</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$Z$</td>
<td>$\omega_1$</td>
</tr>
<tr>
<td>$Z$</td>
<td>0</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>1</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>2</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

Similarly, the rest of the other two rows can be calculated directly:

$$A^* = S^* A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 3 & 1 \end{bmatrix}.$$  

$$b^* = S^* b = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}.$$  

So these rows are:

$\begin{bmatrix} 4 & 0 & 1 & 1 & 8 \\ 3 & 1 & 0 & 1 & 7 \end{bmatrix}.$
Sensitivity Analysis is easy using FI

Fundamental Insight - Sensitivity Analysis

### Initial Tableau:

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Equation Number</th>
<th>Coefficient of</th>
<th>Right side</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>0</td>
<td>$1$ $-20$ $10$</td>
<td>0 0 0 0</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>$0$ $1$ $-1$</td>
<td>1 0 1 7</td>
</tr>
<tr>
<td>$x_2$</td>
<td>2</td>
<td>$0$ $3$ $1$</td>
<td>0 1 7</td>
</tr>
</tbody>
</table>

### Final Tableau:

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Equation Number</th>
<th>Coefficient of</th>
<th>Right side</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>0</td>
<td>$1$ $10$ $0$</td>
<td>0 0 10 70</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>$0$ $4$ $0$</td>
<td>1 1 8</td>
</tr>
<tr>
<td>$x_2$</td>
<td>2</td>
<td>$0$ $3$ $1$</td>
<td>0 1 7</td>
</tr>
</tbody>
</table>

One important application of the **fundamental insight** is to sensitivity analysis. For example, see how easily you can investigate the effect of simultaneously making the following changes in resource levels:

$b_1 = 1 \rightarrow b_1 = 2, \quad \quad b_2 = 7 \rightarrow b_2 = 5.$
Which is better?

### Fundamental Insight - Sensitivity Analysis

#### Initial Tableau:

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Equation Number</th>
<th>Coefficient of</th>
<th>Right side</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z )</td>
<td>0</td>
<td>1 -20 0 0 0</td>
<td>0</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>1</td>
<td>0 1 -1 1 0</td>
<td>1</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>2</td>
<td>0 3 1 0 1</td>
<td>7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Coefficient of</th>
<th>Right side</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -c )</td>
<td>0</td>
</tr>
<tr>
<td>( 0 )</td>
<td>0</td>
</tr>
<tr>
<td>( A )</td>
<td>I</td>
</tr>
<tr>
<td>( b )</td>
<td></td>
</tr>
</tbody>
</table>

#### Final Tableau:

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Equation Number</th>
<th>Coefficient of</th>
<th>Right side</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z )</td>
<td>0</td>
<td>1 0 0 0 10 10 70</td>
<td></td>
</tr>
<tr>
<td>( x_3 )</td>
<td>1</td>
<td>0 4 0 1 1 8</td>
<td></td>
</tr>
<tr>
<td>( x_2 )</td>
<td>2</td>
<td>0 3 1 0 1 7</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Coefficient of</th>
<th>Right side</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^* - c )</td>
<td>( y^* )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>0</td>
</tr>
<tr>
<td>( A^* )</td>
<td>( S^* )</td>
</tr>
<tr>
<td>( b^* )</td>
<td></td>
</tr>
</tbody>
</table>

Old:

\[
Z^* = y^* b = \begin{bmatrix} 0 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 70 \end{bmatrix}.
\]

\[
b^* = S^* b = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 7 \end{bmatrix} = \begin{bmatrix} 8 \end{bmatrix}.
\]

New:

\[
Z^* = y^* b = \begin{bmatrix} 0 & 10 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 50 \end{bmatrix}.
\]

\[
b^* = S^* b = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 5 \end{bmatrix} = \begin{bmatrix} 7 \end{bmatrix}.
\]

This concludes the demonstration. See the OR Tutor menu (to the left) for other demonstrations or close the browser window to exit OR Tutor.
Leveraging FI:

- Revised simplex algorithm (avoid calculating $B^{-1}$ on each Simplex iteration but merely update $B^{-1}$ from iteration to iteration) [See H&L 5.4 for details]

- Allow us to Interpret the shadow prices
  - $(y_1^*, y_2^*, y_3^*, \ldots y_m^*)$
  - $Z^* = y^*b$
  - $Z^* = 0 \times b_1 + 3/2 \times b_2 + 1 \times b_3$

- Enables very efficient postoptimality analysis
  - E.g., modify RHS, b
Put it all together in Tableau and Matrix Form

Initial Tableau Matrix

\[
\begin{bmatrix}
1 & -c & 0 \\
0 & A & I
\end{bmatrix}
\begin{bmatrix}
Z \\
x \\
x_s
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
 b
\end{bmatrix}.
\]

\[x_B = B^{-1}b.\]

\[Z = c_B x_B = c_B B^{-1}b.\]

Notice correspondence with Shadow Prices; a unit of increase in \(b\) contributes \(c_B B^{-1}\) to the objective function; \(c_B B^{-1}\) becomes our shadow price.

**TABLE 5.8 Initial and later simplex tableaux in matrix form**

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basic Variable</th>
<th>Eq.</th>
<th>Coefficient of:</th>
<th>Right Side</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>(Z) Original Variables (\text{Slack Variables})</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>(Z) (x_B)</td>
<td>(0)</td>
<td>1 (-c)</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0) (A)</td>
<td>(0)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0) (I)</td>
<td>(b)</td>
</tr>
<tr>
<td>Any</td>
<td>(Z) (x_B)</td>
<td>(0)</td>
<td>1 (c_B B^{-1} A - c)</td>
<td>(Y) (c_B B^{-1} b)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0) (B^{-1} A)</td>
<td>(c_B B^{-1} b)</td>
</tr>
</tbody>
</table>

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Shadow Prices

Initial Matrix

\[
\begin{bmatrix}
1 & -c & 0 \\
0 & A & I \\
\end{bmatrix}
\begin{bmatrix}
Z \\
x \\
x_s \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
b \\
\end{bmatrix}.
\]

\[x_B = B^{-1}b.\]

\[Z = c_B x_B = c_B B^{-1}b.\]

**TABLE 5.8 Initial and later simplex tableaux in matrix form**

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basic Variable</th>
<th>Eq.</th>
<th>Coefficient of:</th>
<th>Right Side</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Z</td>
<td>Original Variables</td>
</tr>
<tr>
<td>0</td>
<td>(Z) (x_B)</td>
<td>(0)</td>
<td>1 (-c)</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1, 2, \ldots, m)</td>
<td>0 (A)</td>
<td>1</td>
</tr>
<tr>
<td>Any</td>
<td>(Z) (x_B)</td>
<td>(0)</td>
<td>1 (c_B B^{-1} A - c)</td>
<td>(c_B B^{-1} b)</td>
</tr>
</tbody>
</table>
Optimal value; and if optimal BF solution changes (and remains feasible)

By using the formulas,
\[ x_B = S^* b \]
\[ Z^* = y^* b, \]

you can see exactly how the optimal BF solution changes (or whether it becomes infeasible because of negative variables), as well as how the optimal value of the objective function changes, as a function of \( b \). You do not have to reapply the simplex method over and over for each new \( b \), because the coefficients of the slack variables tell all!

For example, consider the change from \( b_2 = 12 \) to \( b_2 = 13 \) as illustrated in Fig. 4.8 for the Wyndor Glass Co. problem. It is not necessary to solve for the new optimal solution \( (x_1, x_2) = \left( \frac{5}{3}, \frac{13}{2} \right) \) because the values of the basic variables in the final tableau \( (b^*) \) are immediately revealed by the fundamental insight:

\[
\begin{bmatrix}
  x_3 \\
  x_2 \\
  x_1
\end{bmatrix}
= b^* = S^* b
= \begin{bmatrix}
  1 & \frac{1}{3} & -\frac{1}{3} \\
  0 & \frac{1}{2} & 0 \\
  0 & -\frac{1}{3} & \frac{1}{3}
\end{bmatrix}
\begin{bmatrix}
  4 \\
  13 \\
  18
\end{bmatrix}
= \begin{bmatrix}
  7 \\
  \frac{13}{3} \\
  \frac{5}{3}
\end{bmatrix}.
\]

There is an even easier way to make this calculation. Since the only change is in the second component of \( b \) (\( \Delta b_2 = 1 \)), which gets premultiplied by only the second column of \( S^* \), the change in \( b^* \) can be calculated as simply

\[
\Delta b^* = \begin{bmatrix}
  \frac{1}{3} \\
  \frac{1}{2} \\
  -\frac{1}{3}
\end{bmatrix}
\Delta b_2
= \begin{bmatrix}
  \frac{1}{3} \\
  \frac{1}{2} \\
  -\frac{1}{3}
\end{bmatrix},
\]

Even more efficient way!
Summary: Simplex in Matrix Form is Fast!

- Matrix operations are a faster way of combining and executing elementary algebraic operations or row operations. Therefore, by using the matrix form of the simplex method, the revised simplex method provides an effective way of adapting the simplex method for computer implementation by focusing on updating $B^{-1}$ (the inverse of the basis matrix).

- The final simplex tableau includes complete information on how it can be algebraically reconstructed directly from the initial simplex tableau.

- This fundamental insight has some very important applications, especially for postoptimality analysis.
Next Lecture: Duality versus Primal

**Primal Problem**

Maximize \[ Z = \sum_{j=1}^{n} c_j x_j, \]
subject to
\[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i = 1, 2, \ldots, m \]
and
\[ x_j \geq 0, \quad \text{for } j = 1, 2, \ldots, n. \]

**Dual Problem**

Minimize \[ W = \sum_{i=1}^{m} b_i y_i, \]
subject to
\[ \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad \text{for } j = 1, 2, \ldots, n \]
and
\[ y_i \geq 0, \quad \text{for } i = 1, 2, \ldots, m. \]

**Primal Problem**

Maximize \[ Z = cx, \]
subject to
\[ Ax \leq b \]
and
\[ x \geq 0. \]

**Dual Problem**

Minimize \[ W = yb, \]
subject to
\[ yA \geq c \]
and
\[ y \geq 0. \]
TABLE 6.1 Primal and dual problems for the Wyndor Glass Co. example

<table>
<thead>
<tr>
<th>Primal Problem in Algebraic Form</th>
<th>Dual Problem in Algebraic Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximize $Z = 3x_1 + 5x_2$, subject to $x_1 \leq 4$</td>
<td>Minimize $W = 4y_1 + 12y_2 + 18y_3$, subject to $y_1 + 3y_3 \geq 3$</td>
</tr>
<tr>
<td>$2x_2 \leq 12$</td>
<td>$2y_2 + 2y_3 \geq 5$</td>
</tr>
<tr>
<td>$3x_1 + 2x_2 \leq 18$ and $x_1 \geq 0, x_2 \geq 0$.</td>
<td>and $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Primal Problem in Matrix Form</th>
<th>Dual Problem in Matrix Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximize $Z = [3, 5]^{\begin{bmatrix} x_1 \ x_2 \end{bmatrix}}$, subject to $\begin{bmatrix} 1 &amp; 0 \ 0 &amp; 2 \ 3 &amp; 2 \end{bmatrix}^{\begin{bmatrix} x_1 \ x_2 \end{bmatrix}} \leq \begin{bmatrix} 4 \ 12 \ 18 \end{bmatrix}$ and $^{\begin{bmatrix} x_1 \ x_2 \end{bmatrix}} \geq \begin{bmatrix} 0 \ 0 \end{bmatrix}$.</td>
<td>Minimize $W = [y_1, y_2, y_3]^{\begin{bmatrix} 1 &amp; 0 \ 0 &amp; 2 \ 3 &amp; 2 \end{bmatrix}} \begin{bmatrix} 4 \ 12 \ 18 \end{bmatrix}$ subject to $^{\begin{bmatrix} y_1, y_2, y_3 \end{bmatrix}} \leq [3, 5]$ and $^{\begin{bmatrix} y_1, y_2, y_3 \end{bmatrix}} \geq [0, 0, 0]$.</td>
</tr>
</tbody>
</table>
• End of lecture
Reading Material

- Read Hillier and Lieberman, pages 1-30 for lecture 1
- For lecture 2, read chapter 3 and Chapter 4, pages 31-107
- For Lecture Chapter 4 and 5
- Explore Simplex on IOR Tutor and OR Tutor
- For Lecture 5, read Chapter 6 and initial parts of Chapter 7.
Guidelines for Homework

• Please provide code, graphs and comments in a Word or PDF report. Don’t forget to put your name, email and date of submission on each report. Please follow the Springer LNCS style (templates for Word and Latex are available at
  – http://www.springer.com/computer/lncs?SGWID=0-164-6-793341-0
  – I.e., pretend you are writing a conference paper (at in format)
• Please provide R code in a separate file .R file and embed the code also in your answers along with the graphs and tables. Please comment your code so that I or anybody else can understand it and please cross reference code with problem numbers and descriptions. Please label each figure and table appropriately.
• Please name files as follows: TIM206-2013-HWK-Week01-
  StudentLastName.R, .doc, .pdf etc..
• Please create a separate driver function for each exercise or exercise part (and comment!) E.g., hw1-Question3.1.1 = function() {.....}
• If you have questions please raise them in class or via email or during office hours if requested
• Homework is due on Wednesday, of the following week by 7PM.
• Please submit your homework by email to: James.Shanahan@gmail.com and Shanahan@soe.ucsc.edu, and jgrahamsf541@gmail.com with the subject “TIM 206 Winter 2013 Homework 3”
• Have fun!
Homework

• Exercises in H&L Book
  - 4.6-1
  - 4.7-1
  - 4.7-3
  - 5.1-1
  - 5.1-4
  - 5.1-9
  - 5.2-1
  - 5.2-2
  - 5.3-1
  - 5.3.2

• HINT: where possible use IOR tutor or R to solve and plot your answers
• End of Homework