Linear Programming
Duality Theory
Dual Simplex, Transportation Problem

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TIM 206 (30155)  Introduction to Optimization Theory and Applications
Thursday, January 31, 2013
Lecture 04
Course Info: Solutions

- [http://courses.soe.ucsc.edu/courses/tim206](http://courses.soe.ucsc.edu/courses/tim206)
- [https://courses.soe.ucsc.edu/courses/tim206/Winter13/01](https://courses.soe.ucsc.edu/courses/tim206/Winter13/01)
  - Schedule
  - Exam during
Performance Evaluation

Final Exam (closed book):
Week 11 of the Quarter

Performance Evaluation:
Homework 30%
Midterm 20% (Week 6 of the Quarter)
Class participation 20%
Final Exam 30% (Week 11 of the Quarter)
Audience Participation
Reading Material

• Chapter 6 and 7 from H&L Book
LP Lecture 4 Schedule

• **Last lecture**
  – Adapting simplex to other forms (\(=, \geq\), negative \(b\))
  – Two Phase, Big-M and Artificial Variable technique.
  – Sensitivity Analysis
  – Shadow Prices
  – Simplex via matrices
  – Fundamental insight

• **This Lecture**
  – Duality Theory
  – Economic interpretation of duality
  – Primal-Dual Relationships
  – The Role of Duality Theory in Sensitivity Analysis
  – Applying Sensitivity Analysis (changes in \(c, b, A\))
  – Dual Simplex
  – Transportation and Assignment problem
  – Case Studies in Digital Advertising
Duality Theory and Sensitivity Analysis

- Duality Theory
- Economic interpretation of duality
- Primal-Dual Relationships
- The Role of Duality Theory in Sensitivity Analysis

- Applying Sensitivity Analysis (changes in c, b, A)
- Sensitivity Analysis in Excel
Duality Theory

- Provides an alternative/dual LP (introduced in 1940s)
- Every linear programming problem has associated with it another linear programming problem called the dual.
- The relationships between the dual problem and the original problem (called the primal) prove to be extremely useful in a variety of ways.
- Duality is useful
  - For example, you soon will see that the shadow prices described actually are provided by the optimal solution for the dual problem.
  - One of the key uses of duality theory lies in the interpretation and implementation of sensitivity analysis.
Duality

• **Dual algorithms**
  – When both LP problems have feasible vectors, they have optimal \( x^* \) and \( y^* \). The minimum cost \( cx^* \) equals the maximum income \( y^*b \). If \( yb = cx \) then \( x \) and \( y \) are optimal. [Duality Theorem]

• **If \( x \) and \( y \) are feasible in the primal and dual problems then \( yb \leq cx \) [weak duality].**

• **Provides a means to conduct sensitivity analysis easily**
  – Resource amounts can be estimates; so maybe want to engage in a what-if analysis
Post Optimality Analysis

• Change in resources (b)
  – Can see how the optimal basic feasible solution changes (or whether it becomes infeasible)
  – The coefficients of the slack variables tell all (so no need to reapply the simplex method all over)
Leveraging FI:

- Revised simplex algorithm (avoid calculating $B^{-1}$ on each Simplex iteration but merely update $B^{-1}$ from iteration to iteration) [See H&L 5.4 for details]

- Allow us to Interpret the shadow prices
  - $(y_1^*, y_2^*, y_3^*, \ldots y_m^*)$
  - $Z^* = y^*b$
  - $Z^* = 0 \times b_1 + 3/2 \times b_2 + 1 \times b_3$

- Enables very efficient postoptimality analysis
  - E.g., modify RHS, b
Summary: Simplex in Matrix Form is Fast!

• **Matrix operations** are a faster way of combining and executing elementary algebraic operations or row operations.
  - Therefore, by using the matrix form of the simplex method, the revised simplex method provides an effective way of adapting the simplex method for computer implementation by focusing on updating $B^{-1}$ (the inverse of the basis matrix).

• **The final simplex tableau includes complete information on how it can be algebraically reconstructed directly from the initial simplex tableau.**

• **This fundamental insight has some very important applications, especially for postoptimality analysis.**
Reading Material

- Explore Simplex on IOR Tutor and OR Tutor
- For Lecture 4, read Chapter 6, Sections 7.1., 8.1, 8.2
LP: Find some feasible solutions

- Primal problem.

\[
\begin{align*}
\text{max} & \quad 4x_1 + x_2 + 5x_3 + 3x_4 \\
\text{s.t.} & \quad x_1 - x_2 - x_3 + 3x_4 \leq 1 \\
& \quad 5x_1 + x_2 + 3x_3 + 8x_4 \leq 55 \\
& \quad -x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3 \\
& \quad x_1, \ x_2, \ x_3, \ x_4 \geq 0
\end{align*}
\]

- Find a lower bound on optimal value.
LP: Find some feasible solutions

• Primal problem.

\[
\begin{align*}
\text{max} & \quad 4x_1 + x_2 + 5x_3 + 3x_4 \\
\text{s.t.} & \quad x_1 - x_2 - x_3 + 3x_4 \leq 1 \\
& \quad 5x_1 + x_2 + 3x_3 + 8x_4 \leq 55 \\
& \quad -x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3 \\
& \quad x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

• Find a lower bound on optimal value.

- \((x_1, x_2, x_3, x_4) = (0, 0, 1, 0)\) \quad \Rightarrow \quad z^* \geq 5.
- \((x_1, x_2, x_3, x_4) = (2, 1, 1, 1/3)\) \quad \Rightarrow \quad z^* \geq 15.
- \((x_1, x_2, x_3, x_4) = (3, 0, 2, 0)\) \quad \Rightarrow \quad z^* \geq 22.
- \((x_1, x_2, x_3, x_4) = (0, 14, 0, 5)\) \quad \Rightarrow \quad z^* \geq 29.
Find an upper bound for the maximum

• Primal problem.

\[
\begin{align*}
\text{max} & \quad 4x_1 + x_2 + 5x_3 + 3x_4 \\
\text{s. t.} & \quad x_1 - x_2 - x_3 + 3x_4 \leq 1 \\
& \quad 5x_1 + x_2 + 3x_3 + 8x_4 \leq 55 \\
& \quad -x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3 \\
& \quad x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

• Find an upper bound on optimal value
  – The obvious value is infinity because it is a maximization problem (if we ignore both constraints, the value will be infinity)
Find an upper bound for the maximum

- **Primal problem.**
  
  $\begin{align*}
  \text{max} & \quad 4x_1 + x_2 + 5x_3 + 3x_4 \\
  \text{s.t.} & \quad x_1 - x_2 - x_3 + 3x_4 \leq 1 \\
  & \quad 5x_1 + x_2 + 3x_3 + 8x_4 \leq 55 \\
  & \quad -x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3 \\
  & \quad x_1, x_2, x_3, x_4 \geq 0
  \end{align*}$

- **Find an upper bound on optimal value.**
  - Multiply $2^{nd}$ inequality by 2: $10x_1 + 2x_2 + 6x_3 + 16x_4 \leq 110$.
  - The optimal solution has to be feasible $X_i \geq 0$ and should satisfy
    - $10x_1 + 2x_2 + 6x_3 + 16x_4 \leq 110$ ($2^{nd}$ inequality by 2)
  
  $\Rightarrow \quad z^* = 4x_1 + x_2 + 5x_3 + 3x_4 \leq 10x_1 + 2x_2 + 6x_3 + 16x_4 \leq 110.$
Find an upper bound for the maximum

- **Primal problem.**

  \[
  \begin{align*}
  \text{max} & \quad 4x_1 + x_2 + 5x_3 + 3x_4 \\
  \text{s. t.} & \quad x_1 - x_2 - x_3 + 3x_4 \leq 1 \\
  & \quad 5x_1 + x_2 + 3x_3 + 8x_4 \leq 55 \\
  & \quad -x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3 \\
  & \quad x_1, x_2, x_3, x_4 \geq 0
  \end{align*}
  \]

  This can be justified only if the coefficients of at each \(x_i\) is at least as big as the correspond coefficient in \(z\)

- **Find an upper bound on optimal value.**
  - Multiply 2\(^{nd}\) inequality by 2: \(10x_1 + 2x_2 + 6x_3 + 16x_4 \leq 110\).
  - The optimal solution has to be feasible \(X_i \geq 0\) and should satisfy
    - \(10x_1 + 2x_2 + 6x_3 + 16x_4 \leq 110\) (2\(^{nd}\) inequality by 2)

  \[\Rightarrow \quad z^* = 4x_1 + x_2 + 5x_3 + 3x_4 \leq 10x_1 + 2x_2 + 6x_3 + 16x_4 \leq 110.\]
UB for Maximum: Add Constraints

- **Primal problem.**
  
  \[
  \begin{align*}
  \text{max} & \quad 4x_1 + x_2 + 5x_3 + 3x_4 \\
  \text{s.t.} & \quad x_1 - x_2 - x_3 + 3x_4 \leq 1 \\
  & \quad 5x_1 + x_2 + 3x_3 + 8x_4 \leq 55 \\
  & \quad -x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3 \\
  & \quad x_1, x_2, x_3, x_4 \geq 0
  \end{align*}
  \]

- **Find an upper bound on optimal value.**
  - Multiply 2\(^{nd}\) inequality by 2: \(10x_1 + 2x_2 + 6x_3 + 16x_4 \leq 110\).

  \[z^* \leq 110 \quad \Rightarrow \quad z^* = 4x_1 + x_2 + 5x_3 + 3x_4 \leq 10x_1 + 2x_2 + 6x_3 + 16x_4 \leq 110.\]

  - Adding 2\(^{nd}\) and 3\(^{rd}\) inequalities: \(4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58.\)

  \[z^* \leq 58 \quad \Rightarrow \quad z^* = 4x_1 + x_2 + 5x_3 + 3x_4 \leq 4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58.\]
LP: add multiples of constraints

• Primal problem.

\[
\begin{align*}
\text{max} & \quad 4x_1 + x_2 + 5x_3 + 3x_4 \\
\text{s. t.} & \quad x_1 - x_2 - x_3 + 3x_4 \leq 1 \\
& \quad 5x_1 + x_2 + 3x_3 + 8x_4 \leq 55 \\
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& \quad x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

• Find an upper bound on optimal value.
  – Adding 11 times 1\textsuperscript{st} inequality to 6 times 3\textsuperscript{rd} inequality:

\[z^* \leq 29 \quad \Rightarrow \quad z^* = 4x_1 + x_2 + 5x_3 + 3x_4 \leq 5x_1 + x_2 + 7x_3 + 3x_4 \leq 29.\]

• Recall.
  – \((x_1, x_2, x_3, x_4) = (0, 14, 0, 5) \Rightarrow z^* \geq 29.\]
LP Duality

- **Primal problem.**

\[
\begin{align*}
\text{max} & \quad 4x_1 + x_2 + 5x_3 + 3x_4 \\
\text{s. t.} & \quad x_1 - x_2 - x_3 + 3x_4 \leq 1 \\
& \quad 5x_1 + x_2 + 3x_3 + 8x_4 \leq 55 \\
& \quad -x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3 \\
& \quad x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

- **General idea:** add linear combination \((y_1, y_2, y_3)\) of the constraints.

\[
\begin{align*}
(y_1 + 5y_2 - y_3)x_1 &+ (-y_1 + y_2 + 2y_3)x_2 + \\
(-y_1 + 3y_2 + 3y_3)x_3 &+ (3y_1 + 8y_2 - 5y_3)x_4 \\
\leq y_1 + 55y_2 + 3y_3
\end{align*}
\]

- Each of these multipliers must be positive otherwise the corresponding inequality would reverse its direction.

\[
4x_1 + x_2 + 5x_3 + 3x_4 \leq \\
(y_1 + 5y_2 - y_3)x_1 + (-y_1 + y_2 + 2y_3)x_2 + \\
(-y_1 + 3y_2 + 3y_3)x_3 + (3y_1 + 8y_2 - 5y_3)x_4 \\
\leq y_1 + 55y_2 + 3y_3
\]

This can be justified only if the coefficients of at each \(x_i\) is at least as big as the corresponding coefficient in \(z\).
LP Duality

- Primal problem.

\[
\begin{align*}
\text{max} \quad & 4x_1 + x_2 + 5x_3 + 3x_4 \\
\text{s.t.} \quad & x_1 - x_2 - x_3 + 3x_4 \leq 1 \\
& 5x_1 + x_2 + 3x_3 + 8x_4 \leq 55 \\
& -x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3 \\
& x_1, \quad x_2, \quad x_3, \quad x_4 \geq 0
\end{align*}
\]

- General idea: add linear combination \((y_1, y_2, y_3)\) of the constraints.

\[
4x_1 + x_2 + 5x_3 + 3x_4 \leq (y_1 + 5y_2 - y_3) x_1 + (-y_1 + y_2 + 2y_3) x_2 + (-y_1 + 3y_2 + 3y_3) x_3 + (3y_1 + 8y_2 - 5y_3) x_4
\]

- Only if:

\[
\begin{align*}
\text{s.t.} \quad & y_1 + 5y_2 - y_3 \geq 4 \\
& -y_1 + y_2 + 2y_3 \geq 1 \\
& -y_1 + 3y_2 + 3y_3 \geq 5 \\
& 3y_1 + 8y_2 - 5y_3 \geq 3 \\
& y_1, \quad y_2, \quad y_3 \geq 0
\end{align*}
\]
LP Duality

- Primal problem.

\[
\begin{align*}
\text{max} & \quad 4x_1 + x_2 + 5x_3 + 3x_4 \\
\text{s.t.} & \quad x_1 - x_2 - x_3 + 3x_4 \leq 1 \\
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& \quad x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

- If this true

\[
\begin{align*}
\text{s.t.} & \quad y_1 + 5y_2 - y_3 \geq 4 \\
& \quad -y_1 + y_2 + 2y_3 \geq 1 \\
& \quad -y_1 + 3y_2 + 3y_3 \geq 5 \\
& \quad 3y_1 + 8y_2 - 5y_3 \geq 3 \\
& \quad y_1, y_2, y_3 \geq 0
\end{align*}
\]

- Then we can conclude

\[
4x_1 + x_2 + 5x_3 + 3x_4 \leq y_1 + 55y_2 + 3y_3
\]
LP Duality

- **Primal problem.**

  \[
  \begin{align*}
  \text{max} & \quad 4x_1 + x_2 + 5x_3 + 3x_4 \\
  \text{s.t.} & \quad x_1 - x_2 - x_3 + 3x_4 \leq 1 \\
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  \]

- **If this true**

  \[
  \begin{align*}
  \text{s.t.} & \quad y_1 + 5y_2 - y_3 \geq 4 \\
  & \quad -y_1 + y_2 + 2y_3 \geq 1 \\
  & \quad -y_1 + 3y_2 + 3y_3 \geq 5 \\
  & \quad 3y_1 + 8y_2 - 5y_3 \geq 3 \\
  & \quad y_1, y_2, y_3 \geq 0
  \end{align*}
  \]

- **Then we can conclude**

  \[ z^* \leq y_1 + 55y_2 + 3y_3 \]
LP Duality

• Primal problem.

\[
\begin{align*}
\text{max } & \quad 4x_1 + x_2 + 5x_3 + 3x_4 \\
\text{s.t. } & \quad x_1 - x_2 - x_3 + 3x_4 \leq 1 \\
& \quad 5x_1 + x_2 + 3x_3 + 8x_4 \leq 55 \\
& \quad -x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3 \\
& \quad x_1, \quad x_2, \quad x_3, \quad x_4 \geq 0
\end{align*}
\]

\(Y_1\) \(Y_2\) \(Y_3\)

• General idea: add linear combination \((y_1, y_2, y_3)\) of the constraints.

\[
(y_1 + 5y_2 - y_3) x_1 + (-y_1 + y_2 + 2y_3) x_2 + \\
(-y_1 + 3y_2 + 3y_3) x_3 + (3y_1 + 8y_2 - 5y_3) x_4 \leq y_1 + 55y_2 + 3y_3
\]

• Dual problem.

\[
\begin{align*}
\text{min } & \quad y_1 + 55y_2 + 3y_3 \\
\text{s.t. } & \quad y_1 + 5y_2 - y_3 \geq 4 \\
& \quad -y_1 + y_2 + 2y_3 \geq 1 \\
& \quad -y_1 + 3y_2 + 3y_3 \geq 5 \\
& \quad 3y_1 + 8y_2 - 5y_3 \geq 3 \\
& \quad y_1, \quad y_2, \quad y_3 \geq 0
\end{align*}
\]
LP Duality

- Primal and dual linear programs: given rational numbers $a_{ij}, b_i, c_j$, find values $x_i, y_j$ that optimize (P) and (D).

\begin{align*}
(P) \quad \text{max} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad 1 \leq i \leq m \\
& \quad x_j \geq 0 \quad 1 \leq j \leq n \\
\end{align*}

\begin{align*}
(D) \quad \text{min} & \quad \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} & \quad \sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad 1 \leq j \leq n \\
& \quad y_i \geq 0 \quad 1 \leq i \leq m \\
\end{align*}

- Duality Theorem (Gale-Kuhn-Tucker 1951, Dantzig-von Neumann 1947). If (P) and (D) are nonempty then $\text{max} = \text{min}$.
  - Dual solution provides certificate of optimality $\Rightarrow$
    decision version
Duality versus Primal

**Primal Problem**

Maximize \( Z = \sum_{j=1}^{n} c_j x_j \),

subject to

\[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i = 1, 2, \ldots, m \]

and

\( x_j \geq 0, \quad \text{for } j = 1, 2, \ldots, n. \)

**Dual Problem**

Minimize \( W = \sum_{i=1}^{m} b_i y_i \),

subject to

\[ \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad \text{for } j = 1, 2, \ldots, n \]

and

\( y_i \geq 0, \quad \text{for } i = 1, 2, \ldots, m. \)

**Primal Problem**

Maximize \( Z = cx \),

subject to

\( Ax \leq b \)

and

\( x \geq 0. \)

**Dual Problem**

Minimize \( W = yb \),

subject to

\( yA \geq c \)

and

\( y \geq 0. \)
### TABLE 6.1 Primal and dual problems for the Wyndor Glass Co. example

<table>
<thead>
<tr>
<th>Primal Problem in Algebraic Form</th>
<th>Dual Problem in Algebraic Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximize $Z = 3x_1 + 5x_2$, subject to $x_1 \leq 4$ \hspace{1cm} $2x_2 \leq 12$ \hspace{1cm} $3x_1 + 2x_2 \leq 18$ \hspace{1cm} and $x_1 \geq 0$, $x_2 \geq 0$.</td>
<td>Minimize $W = 4y_1 + 12y_2 + 18y_3$, subject to $y_1 + 3y_3 \geq 3$ \hspace{1cm} $2y_2 + 2y_3 \geq 5$ \hspace{1cm} and $y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Primal Problem in Matrix Form</th>
<th>Dual Problem in Matrix Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximize $Z = [3, 5]^{\begin{bmatrix} x_1 \ x_2 \end{bmatrix}}$, subject to $\begin{bmatrix} 1 &amp; 0 \ 0 &amp; 2 \ 3 &amp; 2 \end{bmatrix}^{\begin{bmatrix} x_1 \ x_2 \end{bmatrix}} \leq \begin{bmatrix} 4 \ 12 \ 18 \end{bmatrix}$ \hspace{1cm} and $\begin{bmatrix} x_1 \ x_2 \end{bmatrix} \geq \begin{bmatrix} 0 \ 0 \end{bmatrix}$.</td>
<td>Minimize $W = [y_1, y_2, y_3]^{\begin{bmatrix} 4 \ 12 \ 18 \end{bmatrix}}$, subject to $\begin{bmatrix} 1 &amp; 0 \ 0 &amp; 2 \ 3 &amp; 2 \end{bmatrix}^{\begin{bmatrix} y_1, y_2, y_3 \end{bmatrix}} \geq \begin{bmatrix} 3, 5 \end{bmatrix}$ \hspace{1cm} and $\begin{bmatrix} y_1, y_2, y_3 \end{bmatrix} \geq \begin{bmatrix} 0, 0, 0 \end{bmatrix}$.</td>
</tr>
</tbody>
</table>
TABLE 6.2 Primal-dual table for linear programming, illustrated by the Wyndor Glass Co. example

(a) General Case

<table>
<thead>
<tr>
<th>Dual Problem</th>
<th>Primal Problem</th>
<th>Right Side</th>
<th>Coefficients for Objective Function (Minimize)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coefficient of:</td>
<td>Coefficient of:</td>
<td>Right Side</td>
<td></td>
</tr>
<tr>
<td>$y_1$</td>
<td>$a_{11}$ $a_{12}$ $\ldots$ $a_{1n}$</td>
<td>$\leq b_1$</td>
<td></td>
</tr>
<tr>
<td>$y_2$</td>
<td>$a_{21}$ $a_{22}$ $\ldots$ $a_{2n}$</td>
<td>$\leq b_2$</td>
<td></td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td></td>
</tr>
<tr>
<td>$y_m$</td>
<td>$a_{m1}$ $a_{m2}$ $\ldots$ $a_{mn}$</td>
<td>$\leq b_m$</td>
<td></td>
</tr>
</tbody>
</table>

(b) Wyndor Glass Co. Example

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>1</td>
<td>0</td>
<td>$\leq 4$</td>
</tr>
<tr>
<td>$y_2$</td>
<td>0</td>
<td>2</td>
<td>$\leq 12$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>3</td>
<td>2</td>
<td>$\leq 18$</td>
</tr>
</tbody>
</table>

| VI | VI | 3 | 5 |
General relationship between Dual and Primal

- Duality theory is based directly on the fundamental insight (particularly with regard to row 0). To see why, we continue to use the notation introduced for row 0 of the final tableau.
Weak Duality

• If \( x \) is a feasible solution for the primal problem and \( y \) is a feasible solution for the dual problem, then
\[
  cx \leq yb.
\]
  – For example, for the Wyndor Glass Co. problem, one feasible solution is \( x_1 = 3, x_2 = 3 \), which yields \( Z = cx = 24 \), and one feasible solution for the dual problem is \( y_1 = 1, y_2 = 1, y_3 = 2 \), which yields a larger objective function value \( W = yb = 52 \).
  – These are just sample feasible solutions for the two problems.

• For any such pair of feasible solutions, this inequality must hold because the maximum feasible value of \( Z = cx \) (36) equals the minimum feasible value of the dual objective function \( W = yb \), which is our next property.
Weak Duality Theorem

• This theorem is very useful
• Suppose there is a feasible solution \( y \) to \( D \). Then any feasible solution of \( P \) has value upper bounded by \( b^T y \). This means that if \( P \) has a feasible solution, then it has an optimal solution
• Reversing argument is also true
• Therefore, if both \( P \) and \( D \) have feasible solutions, then both must have an optimal solution.
Strong duality property

- If \( x^* \) is an optimal solution for the primal problem and \( y^* \) is an optimal solution for the dual problem, then
  - \( cx^* = y^* b \).

- Thus, these two properties imply that \( cx < yb \) for feasible solutions if one or both of them are not optimal for their respective problems, whereas equality holds when both are optimal.
Each iteration of the Simplex

• The weak duality property describes the relationship between any pair of solutions for the primal and dual problems where both solutions are feasible for their respective problems.

• At each iteration, the simplex method finds a specific pair of solutions for the two problems, where the primal solution is feasible but the dual solution is not feasible (except at the final iteration).

• Complementary Basic Solutions
  – Our next property describes this situation and the relationship between this pair of solutions.
Complementary solutions property

- At each iteration, the simplex method simultaneously identifies a CPF solution $x$ for the primal problem and a complementary solution $y$ for the dual problem (found in row 0, the coefficients of the slack variables), where $cx = yb$.

- If $x$ is not optimal for the primal problem, then $y$ is not feasible for the dual problem.
Final Tableau in Full Matrix Form

Complementary Solutions

Basic Feasible Solution in primal

If \( x > 0 \) then Basic variable
Then corresponding dual variable is 0
\( \Rightarrow \) Variable will be nonbasic in dual problem
Complementary solutions property

- To illustrate, after one iteration for the Wyndor Glass Co. problem, \(x_1 = 0, x_2 = 6,\) and \(y_1 = 0, y_2 = \frac{5}{2}, y_3 = 0,\) with \(cx = 30 = yb.\)
- This \(x\) is feasible for the primal problem, but this \(y\) is not feasible for the dual problem (since it violates the constraint, \(y_1 + 3y_3 \geq 3\)).
- The complementary solutions property also holds at the final iteration of the simplex method, where an optimal solution is found for the primal problem.

### TABLE 6.1 Primal and dual problems for the Wyndor Glass Co. example

**Primal Problem in Algebraic Form**

Maximize \(Z = 3x_1 + 5x_2,\)

subject to

\[
\begin{align*}
    x_1 &\leq 4 \\
    2x_2 &\leq 12 \\
    3x_1 + 2x_2 &\leq 18 \\
\end{align*}
\]

and \(x_1 \geq 0, \quad x_2 \geq 0.\)

**Dual Problem in Algebraic Form**

Minimize \(W = 4y_1 + 12y_2 + 18y_3,\)

subject to

\[
\begin{align*}
    y_1 + 3y_3 &\geq 3 \\
    2y_2 + 2y_3 &\geq 5 \\
\end{align*}
\]

and \(y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0.\)
WynDor Number of possible solutions

- \( n+m \) choose \( m = 2+3 \) Choose 3 = 10 primal solutions
- \( m+n \) choose \( n = 10 \) dual solutions
- 10 Complementary solutions
Complementary optimal solutions property

- At the final iteration, the simplex method simultaneously identifies an optimal solution \( x^* \) for the primal problem and a complementary optimal solution \( y^* \) for the dual problem (found in row 0, the coefficients of the slack variables), where
  - \( cx^* = y^* b \).

- The \( y_i^* \) are the shadow prices for the primal problem.
- For the example, the final iteration yields \( x_1^* = 2, x_2^* = 6, \) and \( y_1^* = 0, y_2^* = 3/2, y_3^* = 1, \)
- with \( cx^* = 36 = y^* b \).
Surplus/slack vars

TABLE 6.5 Row 0 and corresponding dual solution for each iteration for the Wyndor Glass Co. example

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Primal Problem</th>
<th>Dual Problem</th>
<th>W</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><strong>Row 0</strong></td>
<td><strong>y₁, y₂, y₃</strong></td>
<td><strong>z₁ - c₁, z₂ - c₂</strong></td>
</tr>
<tr>
<td>0</td>
<td>[-3, -5</td>
<td>0, 0, 0, 0</td>
<td>0]</td>
</tr>
<tr>
<td>1</td>
<td>[-3, 0, 0, 5/2, 0, 30]</td>
<td>0</td>
<td>5/2</td>
</tr>
<tr>
<td>2</td>
<td>[0, 0, 0, 3/2, 1, 36]</td>
<td>0</td>
<td>3/2</td>
</tr>
</tbody>
</table>

y₁ + 3y₃ ≥ 3 is violated

Complementary solutions property
Symmetry Property

- For any primal problem and its dual problem, all relationships between them must be symmetric because the dual of this dual problem is this primal problem.

- Consequently, the simplex method can be applied to either problem, and it simultaneously will identify complementary solutions (ultimately a complementary optimal solution) for the other problem.
Complementary Basic Solutions

• One of the important relationships between the primal and dual problems is a direct correspondence between their basic solutions.
• The key to this correspondence is row 0 of the simplex tableau for the primal basic solution.
• A complete solution for the dual problem (including the surplus variables) can be read directly from row 0.
• Because of its coefficient in row 0, each variable in the primal problem has an associated variable in the dual problem.
Final Tableau in Full Matrix Form

Complementary Solutions

Basic Feasible Solution in primal

If \( x > 0 \) then Basic variable
Then corresponding dual variable is 0
\( \Rightarrow \) Variable will be nonbasic in dual problem
Example. To illustrate this matrix form for the current set of equations, we will show how it yields the final set of equations resulting from iteration 2 for the Wyndor Glass Co. problem. Using the $B^{-1}$ and $c_B$ given for iteration 2 at the end of the preceding subsection, we have

\[
B^{-1}A = \begin{bmatrix}
1 & \frac{1}{3} & -\frac{1}{3} \\
0 & \frac{1}{2} & 0 \\
0 & -\frac{1}{3} & \frac{1}{3}
\end{bmatrix}\begin{bmatrix}
1 \\
0 \\
3
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix},
\]

\[
c_BB^{-1} = \begin{bmatrix}
1 & \frac{1}{3} & -\frac{1}{3} \\
0 & \frac{1}{2} & 0 \\
0 & -\frac{1}{3} & \frac{1}{3}
\end{bmatrix}\begin{bmatrix}
0, 5, 3 \\
0, 2, 1
\end{bmatrix} = \begin{bmatrix}
0, \frac{3}{2}, 1
\end{bmatrix},
\]

\[
c_BB^{-1}A - c = \begin{bmatrix}
0 & 0 \\
0 & 1 \\
1 & 0
\end{bmatrix} - [3, 5] = [0, 0].
\]

Also, by using the values of $x_B = B^{-1}b$ and $Z = c_BB^{-1}b$ calculated at the end of the preceding subsection, these results give the following set of equations:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}\begin{bmatrix}
Z \\
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
36 \\
2 \\
6
\end{bmatrix},
\]

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}\begin{bmatrix}
Z \\
x_3 \\
x_4
\end{bmatrix} = \begin{bmatrix}
2 \\
6 \\
2
\end{bmatrix},
\]

as shown in the final simplex tableau in Table 4.8.
Final Tableau in Full Matrix Form

Complementary Solutions

Basic Feasible Solution in dual (why feasible in the dual?)

Basic Feasible Solution in primal

If $x > 0$ then Basic variable
Then corresponding dual variable is 0
⇒ Variable will be nonbasic in dual problem
### TABLE 6.7 Association between variables in primal and dual problems

<table>
<thead>
<tr>
<th>Any problem</th>
<th>Primal Variable</th>
<th>Associated Dual Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(Decision variable) $x_j$</td>
<td>$z_j - c_j$ (surplus variable) $j = 1, 2, \ldots, n$</td>
</tr>
<tr>
<td></td>
<td>(Slack variable) $x_{n+1}$</td>
<td>$y_i$ (decision variable) $i = 1, 2, \ldots, m$</td>
</tr>
<tr>
<td>Wyndor problem</td>
<td>Decision variables: $x_1, x_2$</td>
<td>$z_1 - c_1$ (surplus variables)</td>
</tr>
<tr>
<td></td>
<td>Slack variables: $x_3, x_4, x_5$</td>
<td>$z_2 - c_2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y_1$ (decision variables)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y_2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y_3$</td>
</tr>
</tbody>
</table>

### TABLE 6.8 Complementary slackness relationship for complementary basic solutions

<table>
<thead>
<tr>
<th>Primal Variable</th>
<th>Associated Dual Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basic</td>
<td>Nonbasic $(m$ variables)</td>
</tr>
<tr>
<td>Nonbasic</td>
<td>Basic $(n$ variables)</td>
</tr>
</tbody>
</table>
Row 0 in Primal becomes a Basic solution in Dual
Complementary Basic Solutions

• Each basic solution in the primal problem has a complementary basic solution in the dual problem, where their respective objective function values (Z and W) are equal.

• Given row 0 of the simplex tableau for the primal basic solution, the complementary dual basic solution (y, z -c) is found.
Primal Dual Correspondence

**TABLE 6.5** Row 0 and corresponding dual solution for each iteration for the Wyndor Glass Co. example

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Primal Problem</th>
<th>Dual Problem</th>
<th>$W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$[-3, -5, 0, 0, 0, 0, 0]$</td>
<td>$y_1, y_2, y_3$</td>
<td>$z_1 - c_1, z_2 - c_2$</td>
</tr>
<tr>
<td>1</td>
<td>$[-3, 0, 0, \frac{5}{2}, 0, 0]$</td>
<td>$0, \frac{5}{2}, 0$</td>
<td>$-3, 0$</td>
</tr>
<tr>
<td>2</td>
<td>$[0, 0, 0, \frac{3}{2}, 1]$</td>
<td>$0, \frac{3}{2}, 1$</td>
<td>0, 0</td>
</tr>
</tbody>
</table>
Complementary Slackness

- Given the association between the variables in the primal basic solution and the complementary dual basic solution satisfy the complementary slackness relationship shown in

  \[ y_j x_j = 0 \]  \hspace{0.5cm} (where x is the basic solution and y is the dual)

- Furthermore, this relationship is a symmetric one, so that these two basic solutions are complementary to each other.

- \[ y_j x_j = 0 \] (where x is the basic solution and y is the dual)
### TABLE 6.7 Association between variables in primal and dual problems

<table>
<thead>
<tr>
<th>Any problem</th>
<th>Primal Variable</th>
<th>Associated Dual Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(Decision variable) $x_j$</td>
<td>$z_j - c_j$ (surplus variable) $j = 1, 2, \ldots, n$</td>
</tr>
<tr>
<td></td>
<td>(Slack variable) $x_{n+1}$</td>
<td>$y_i$ (decision variable) $i = 1, 2, \ldots, m$</td>
</tr>
</tbody>
</table>

| Wyndor problem | Decision variables: $x_1$ $x_2$ |
|               | Slack variables: $x_3$ $x_4$ $x_5$ |
|               | $z_1 - c_1$ (surplus variables) $z_2 - c_2$ |
|               | $y_1$ (decision variables) $y_2$ $y_3$ |

### TABLE 6.8 Complementary slackness relationship for complementary basic solutions

<table>
<thead>
<tr>
<th>Primal Variable</th>
<th>Associated Dual Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basic</td>
<td>Nonbasic ($m$ variables)</td>
</tr>
<tr>
<td>Nonbasic</td>
<td>Basic ($n$ variables)</td>
</tr>
</tbody>
</table>
Complementary Slackness

• Theorem:
Let x and y be primal and dual feasible solutions respectively. Then x and y are both optimal iff two of the following conditions are satisfied:

• \((A^T y - c)_j x_j = 0\) for all \(j = 1\ldots n\) (\(x_j\) is a decision variable in primal and it corresponding dual; and \((A^T y - c)_j\) is the dual)

• \((Ax - b)_i y_i = 0\) for all \(i = 1\ldots m\) (\(y_i\) is decision variable in dual)
Proof of Complementary Slackness

Proof:
As in the proof of the weak duality theorem, we have: 
\[ c^T x \geq (A^T y)^T x = y^T A x \geq y^T b \]  \hspace{1cm} (1)

From the strong duality theorem, we have:

\[ x \text{ and } y \text{ are optimal} \iff c^T x = b^T y \]
\[ \iff c^T x = y^T A x = y^T b \]
\[ \iff (y^T A - c^T)x = 0 \]
\[ \text{and} \quad y^T (b - A x) = 0 \] \hspace{1cm} (2)
\[ \text{and} \quad y^T (b - A x) = 0 \] \hspace{1cm} (3)
Proof (cont)

Note that
\[(y^T A - c^T)x = \sum_{j=1}^{n} (y^T A - c^T)_j x_j = \sum_{j=1}^{n} (A^T y - c)_j x_j \quad (4)\]

and
\[y^T (b - Ax) = \sum_{i=1}^{m} (b - Ax)_i y_i \quad (5)\]

We have:
x and y optimal ⇔ (2) and (3) hold
⇔ both sums (4) and (5) are zero
⇔ all terms in both sums are zero (?)
⇔ Complementary slackness holds
Why do we care?

- It’s an easy way to check whether a pair of primal/dual feasible solutions are optimal
- Given one optimal solution, complementary slackness makes it easy to find the optimal solution of the dual problem
- May provide a simpler way to solve the primal
Final Tableau in Full Matrix Form

Complementary Solutions

Basic Feasible Solution in dual (why feasible?)

TABLE 6.1 Primal and dual problems for the Wyndor Glass Co. example

<table>
<thead>
<tr>
<th>Basic Feasible Solution in Primal</th>
<th>Feasible?</th>
<th>Z = W</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0, 4, 12, 18)</td>
<td>Yes</td>
<td>0</td>
</tr>
<tr>
<td>(4, 0, 12, 6)</td>
<td>Yes</td>
<td>12</td>
</tr>
<tr>
<td>(6, 0, -2, 12, 0)</td>
<td>No</td>
<td>18</td>
</tr>
<tr>
<td>(4, 3, 6, 0)</td>
<td>Yes</td>
<td>27</td>
</tr>
<tr>
<td>(0, 6, 4, 0, 6)</td>
<td>Yes</td>
<td>30</td>
</tr>
<tr>
<td>(2, 6, 2, 0, 0)</td>
<td>Yes</td>
<td>36</td>
</tr>
<tr>
<td>(4, 0, 0, -2, 0)</td>
<td>No</td>
<td>42</td>
</tr>
<tr>
<td>(0, 9, 4, -6, 0)</td>
<td>No</td>
<td>45</td>
</tr>
</tbody>
</table>

TABLE 6.9 Complementary basic solutions for the Wyndor Glass Co. example

<table>
<thead>
<tr>
<th>No.</th>
<th>Basic Solution</th>
<th>Feasible?</th>
<th>Z = W</th>
<th>Dual Problem</th>
<th>Basic Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0, 0, 4, 12, 18)</td>
<td>Yes</td>
<td>0</td>
<td>(0, 0, 0, -3, -5)</td>
<td>(x3, x4, x5, x1, x2)</td>
</tr>
<tr>
<td>2</td>
<td>(4, 0, 12, 6)</td>
<td>Yes</td>
<td>12</td>
<td>(3, 0, 0, 0, -5)</td>
<td>(y1, y2, y3, y4, y5)</td>
</tr>
<tr>
<td>3</td>
<td>(6, 0, -2, 12, 0)</td>
<td>No</td>
<td>18</td>
<td>(0, 0, 1, 0, -3)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(4, 3, 6, 0)</td>
<td>Yes</td>
<td>27</td>
<td>(-9/2, 0, 5/2, 0, 0)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(0, 6, 4, 0, 6)</td>
<td>Yes</td>
<td>30</td>
<td>(0, 5/2, 0, -3, 0)</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>(2, 6, 2, 0, 0)</td>
<td>Yes</td>
<td>36</td>
<td>(0, 3/2, 1, 0, 0)</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>(4, 6, 0, 0, -6)</td>
<td>No</td>
<td>42</td>
<td>(3, 5/2, 0, 0, 0)</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>(0, 9, 4, -6, 0)</td>
<td>No</td>
<td>45</td>
<td>(0, 0, 5/2, 9/2, 0)</td>
<td></td>
</tr>
</tbody>
</table>
Example (see prev slide): Wyndor Glass

- All eight of its basic solutions (five feasible and three infeasible) are shown in Table 6.9. Thus, its dual problem (see Table 6.1) also must have eight basic solutions, each complementary to one of these primal solutions, as shown in Table 6.9.

- The three BF solutions obtained by the simplex method for the primal problem are the first, fifth, and sixth primal solutions shown in Table 6.9.
  - You already saw in Table 6.5 how the complementary basic solutions for the dual problem can be read directly from row 0, starting with the coefficients of the slack variables and then the original variables.

- The other dual basic solutions also could be identified in this way by constructing row 0 for each of the other primal basic solutions.

- For each primal basic solution, the complementary slackness property can be used to identify the basic and nonbasic variables for the complementary dual basic solution, so that the system (see next slide for example of this).
Example: given primal basic solution derive dual basic solution

Primal BS = (4, 6, 0, 0, -6), x1, x2, x5 are basic vars
Implies Dual decision variables are y3, y4 and y5 are nonbasic and zeros


**Table 6.1 Primal and dual problems for the Wyndor Glass Co. example**

<table>
<thead>
<tr>
<th>Primal Problem in Algebraic Form</th>
<th>Dual Problem in Algebraic Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximize $Z = 3x_1 + 5x_2,$ subject to $x_1 \leq 4$</td>
<td>Minimize $W = 4y_1 + 12y_2 + 18y_3,$ subject to $y_1 + 3y_3 \geq 3$</td>
</tr>
<tr>
<td>$2x_2 \leq 12$</td>
<td>$2y_2 + 2y_3 \geq 5$</td>
</tr>
<tr>
<td>$3x_1 + 2x_2 \leq 18$</td>
<td>and $y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0.$</td>
</tr>
<tr>
<td>and $x_1 \geq 0, \quad x_2 \geq 0.$</td>
<td></td>
</tr>
</tbody>
</table>

Primal BS = (4, 6, 0, 0, -6), $x_1, x_2, x_5$ are basic vars

Estimate the Complementary Dual basic solution
Primal BS = (4, 6, 0, 0, -6), x1, x2, x5 are basic vars
Implies Dual decision variables are y3, y4 and y5 are nonbasic and there for zeros

Given that y3=0, y4=0 and y5=0 in the dual space solve for y1 and y2
y1=3
Y2 =5/2

Complementary Dual BS (3, 5/2, 0, 0, 0) to Primal BS
= (4, 6, 0, 0, -6); W = (3×4 + 5/2 ×12 + 0×18)=42
b×y = 36 for dual optimal value W for (0, 3/2, 1, 0,0)
relationships between complementary basic solutions

- Each optimal basic solution in the primal problem has a complementary optimal basic solution in the dual problem, where their respective objective function values \( Z \) and \( W \) are equal. (Use row 0)

**TABLE 6.9 Complementary basic solutions for the Wyndor Glass Co. example**

<table>
<thead>
<tr>
<th>No.</th>
<th>Primal Problem</th>
<th>Feasible?</th>
<th>Dual Problem</th>
<th>Feasible?</th>
<th>Basic Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Basic Solution</td>
<td></td>
<td></td>
<td>Basic Solution</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>(0, 0, 4, 12, 18)</td>
<td>Yes</td>
<td>0</td>
<td>No</td>
<td>(0, 0, 0, -3, -5)</td>
</tr>
<tr>
<td>2</td>
<td>(4, 0, 0, 12, 6)</td>
<td>Yes</td>
<td>12</td>
<td>No</td>
<td>(3, 0, 0, 0, -5)</td>
</tr>
<tr>
<td>3</td>
<td>(6, 0, -2, 12, 0)</td>
<td>No</td>
<td>18</td>
<td>No</td>
<td>(0, 0, 1, 0, -3)</td>
</tr>
<tr>
<td>4</td>
<td>(4, 3, 0, 6, 0)</td>
<td>Yes</td>
<td>27</td>
<td>No</td>
<td>( \left(\frac{9}{2}, 0, \frac{5}{2}, 0, 0\right) )</td>
</tr>
<tr>
<td>5</td>
<td>(0, 6, 4, 0, 6)</td>
<td>Yes</td>
<td>30</td>
<td>No</td>
<td>( \left(0, \frac{5}{2}, 0, -3, 0\right) )</td>
</tr>
<tr>
<td>6</td>
<td>(2, 6, 2, 0, 0)</td>
<td>Yes</td>
<td>36</td>
<td>Yes</td>
<td>( \left(0, \frac{3}{2}, 1, 0, 0\right) )</td>
</tr>
<tr>
<td>7</td>
<td>(4, 6, 0, 0, -6)</td>
<td>No</td>
<td>42</td>
<td>Yes</td>
<td>( \left(3, \frac{5}{2}, 0, 0, 0\right) )</td>
</tr>
<tr>
<td>8</td>
<td>(0, 9, 4, -6, 0)</td>
<td>No</td>
<td>45</td>
<td>Yes</td>
<td>( \left(0, 0, \frac{5}{2}, \frac{9}{2}, 0\right) )</td>
</tr>
</tbody>
</table>
Complementary basic solutions

Maximization

\[ \sum_{j=1}^{n} c_j x_j = Z \]

Minimization

\[ W = \sum_{i=1}^{m} b_i y_i \]

Primal Max LPs
Search here
<table>
<thead>
<tr>
<th>Primal Basic Solution</th>
<th>Complementary Dual Basic Solution</th>
<th>Both Basic Solutions</th>
<th>Primal Feasible?</th>
<th>Dual Feasible?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Suboptimal</td>
<td>Superoptimal</td>
<td>Yes</td>
<td>No</td>
<td></td>
</tr>
<tr>
<td>Optimal</td>
<td>Optimal</td>
<td>Yes</td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>Superoptimal</td>
<td>Suboptimal</td>
<td>No</td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>Neither feasible nor superoptimal</td>
<td>Neither feasible nor superoptimal</td>
<td>No</td>
<td>No</td>
<td></td>
</tr>
</tbody>
</table>
Primal Dual Programs For Maximization Primals

Minimization

Maximization

Dual Program

Primal Program

Primal solutions

Dual solutions

Primal optimal = Dual optimal

Von Neumann [1947]

Primal solutions

Dual solutions
Duality theorem

• The following are the only possible relationships between the primal and dual problems.
  – 1. If one problem has feasible solutions and a bounded objective function (and so has an optimal solution), then so does the other problem, so both the weak and strong duality properties are applicable.
  – 2. If one problem has feasible solutions and an unbounded objective function (and so no optimal solution), then the other problem has no feasible solutions.
  – 3. If one problem has no feasible solutions, then the other problem has either no feasible solutions or an unbounded objective function.
How to leverage complementary solutions property?

• Lots of constraints in the primal ⇒ solve dual as it will have less constraints
  – The number of functional constraints affects the computational effort of the simplex method far more than the number of variables does. If \( m > n \), so that the dual problem has fewer functional constraints \( (n) \) than the primal problem \( (m) \), then applying the simplex method directly to the dual problem instead of the primal problem probably will achieve a substantial reduction in computational effort.

• Evaluate a proposed primal feasible solution \( x \), by looking at complementary solution \( y \) and it \( cx = by \) then optimal

• Dual simplex algorithm
  – One of the key applications of the complementary solutions property is its use in the dual simplex method. This algorithm operates on the primal problem exactly as if the simplex method …

• Sensitivity Analysis
• Break
Economic Interpretation of Duality

- The dual variable $y_i$ is interpreted as the contribution to profit per unit of resource $i$ ($i = 1, 2, \ldots, m$), when the current set of basic variables is used to obtain the primal solution.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basic Variable</th>
<th>Eq.</th>
<th>Coefficient of:</th>
<th>Right Side</th>
</tr>
</thead>
<tbody>
<tr>
<td>Any</td>
<td>$Z$</td>
<td>(0)</td>
<td>$z_1 - c_1$</td>
<td>$W$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$z_2 - c_2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\cdots$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$z_n - c_n$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$y_1$</td>
<td></td>
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<tr>
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<td></td>
<td></td>
<td>$y_2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\cdots$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$y_m$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Primal Problem</th>
<th>Dual Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$[-3, -5]$</td>
<td>$0$</td>
</tr>
<tr>
<td>1</td>
<td>$[-3, 0]$</td>
<td>$0, 0, 30$</td>
</tr>
<tr>
<td>2</td>
<td>$[0, 0]$</td>
<td>$0, 36$</td>
</tr>
</tbody>
</table>
Shadow Prices

• The shadow price for resource i (denoted by \( y_i^* \)) measures the marginal value of this resource, i.e., the rate at which \( Z \) could be increased by (slightly) increasing the amount of this resource (\( b_i \)) being made available.

• The simplex method identifies this shadow price by \( y_i^* \) coefficient of the ith slack variable in row 0 of the final simplex tableau.
Manager explores different level of resources

- **The tentative initial decision has been**
  - $b_1 = 4$, $b_2 = 12$, $b_3 = 18$,

- **Shadow prices**
  - $y_1^* = 0$  shadow price for resource 1,
  - $y_2^* = 1.5$  shadow price for resource 2,
  - $Y_3 = 1$  shadow price for resource 3.
Should this actually be done?

The optimal solution, \((2, 6)\) with \(Z = 36\), changes to \((5/3, 13/2)\) with \(Z = 37.5\) when \(b_2\) is increased by 1 (from 12 to 13) so that:

\[ y^*_2 = \Delta Z = 37.5 - 36 = 3/2 \]
Shadow Prices: Focus on $y_1^*$

**TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem**

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basic Variable</th>
<th>Eq.</th>
<th>Coefficient of:</th>
<th>Right Side</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$Z$</td>
<td>(0)</td>
<td>$Z$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$x_3$</td>
<td>(1)</td>
<td>$x_1$</td>
<td>-3</td>
</tr>
<tr>
<td></td>
<td>$x_4$</td>
<td>(2)</td>
<td>$x_2$</td>
<td>-5</td>
</tr>
<tr>
<td></td>
<td>$x_5$</td>
<td>(3)</td>
<td>$x_3$</td>
<td>10</td>
</tr>
</tbody>
</table>

Define $x_4 = 18 - 3x_1 - 2x_2$.
Maximize $Z = 3x_1 + 5x_2 - Mx_5$.
subject to $x_1 \leq 4$
$2x_2 \leq 12$
$3x_1 + 2x_2 \leq 18$
and $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$
Shadow Prices Graphically: $y_1^*$

Define $x_5 = 18 - 3x_1 - 2x_2$. Maximize $Z = 3x_1 + 5x_2 - Mx_5$, subject to

\[ x_1 \leq 4 \]
\[ 2x_2 \leq 12 \]
\[ 3x_1 + 2x_2 \leq 18 \]

and $x_1 \geq 0$, $x_2 \geq 0$, $x_5 \geq 0$

---

Should this actually be done?

Why is

\[ y_0^* = \Delta Z = 36 - 36 = \frac{3}{2} \]
Shadow Prices Graphically:

$y_1^* = 0$ means what?
What about constraint 3? Binding?

Because the limited supply of these resources ($b_2 = 12$, $b_3 = 18$) binds $Z$ from being increased further, they have positive shadow prices. Economists refer to such resources as **scarce goods** whereas resources available in surplus (such as resource 1) are **free goods** (resources with a zero shadow price). Discuss shadow prices more later.
Sensitivity Analysis: RHS  $b_i$

- LP($a_{ij}, b_i, c_j$)
- Sensitive parameters (i.e., those that cannot be changed without changing the optimal solution)
- Case $b_i$, if $y_i^* > 0$ (non-basic) implies the optimal solution changes if $b_i$ changes ($b_i$ is a sensitive parameter; feasible region expands/contracts)
- Case $b_i$, if $y_i^* = 0$ (basic): not sensitive (to at least small changes in $b_i$)

- Pay attention to resources with large shadow prices
Dual: minimize the resources consumed by the products

- Dual can be viewed as minimizing the total implicit value of the resources consumed by the activities.
- For the Wyndor problem, the total implicit value (in thousands of dollars per week) of the resources consumed by the two products is $W = 4y_1 + 12y_2 + 18y_3$.

\[
\sum_{i=1}^{m} a_{ij}y_i = c_j, \quad \text{if } x_j > 0 \quad (j = 1, 2, \ldots, n),
\]
\[
y_i = 0, \quad \text{if } x_{n+i} > 0 \quad (i = 1, 2, \ldots, m).
\]

- The economic interpretation of the first statement is that whenever an activity $j$ operates at a strictly positive level ($x_j > 0$), the marginal value of the resources it consumes must equal (as opposed to exceeding) the unit profit from this activity.
Shadow Prices

\[ \sum_{i=1}^{m} a_{ij} y_i = c_j, \quad \text{if } x_j > 0 \quad (j = 1, 2, \ldots, n), \]
\[ y_i = 0, \quad \text{if } x_{n+i} > 0 \quad (i = 1, 2, \ldots, m). \]

- The second statement implies that the marginal value of resource \( i \) is zero \( (y_i = 0) \) whenever the supply of this resource is not exhausted by the activities \((x_{n+i} > 0)\).
- In economic terminology, such a resource is a “free good”; the price of goods that are oversupplied must drop to zero by the law of supply and demand.
- This fact is what justifies interpreting the objective for the dual problem as minimizing the total implicit value of the resources consumed, rather than the resources allocated.
Class Exercise: Role of Duality in Sensitivity Analysis

• Sensitivity analysis basically involves investigating the effect on the optimal solution of making changes in the values of the model parameters $a_{ij}$, $b_i$, and $c_j$.

• However, changing parameter values in the primal problem also changes the corresponding values in the dual problem. Therefore, you have your choice of which problem to use to investigate each change.
  – Because of the primal-dual relationships (especially the complementary basic solutions property), it is easy to move back and forth between the two problems as desired.

• In some cases, it is more convenient to analyze the dual problem directly in order to determine the complementary effect on the primal problem.
Two Sample Sensitivity Cases

• **Changes in the Coefficients of a Nonbasic Variable**
  – What is the effect of these changes on this solution? Is it still feasible? Is it still optimal?
  – Page 215

• **Introduction of a New Variable (or product)**
  – Page 216
Changes in the Coefficients of a Nonbasic Variable

• Because the variable involved is nonbasic (value of zero) \((x,\text{ changing its coefficients cannot affect the feasibility of the solution}).

• Therefore, the open question in this case is whether it is still optimal.
  – An equivalent question is whether the complementary basic solution for the dual problem is still feasible after these changes are made.
  – Since these changes affect the dual problem by changing only one constraint, this question can be answered simply by checking whether this complementary basic solution still satisfies this revised constraint.
Select initial product set

- Products/activities were selected from a larger group of possible activities, where the remaining activities were not included in the original model because they seemed less attractive.
  - Or perhaps these other activities did not come to light until after the original model was formulated and solved.
- Either way, the key question is whether any of these previously unconsidered activities are sufficiently worthwhile to warrant initiation.
- In other words, would adding any of these activities to the model change the original optimal solution?
Adding a product change the LP model

- Adding another activity amounts to introducing a new variable, with the appropriate coefficients in the functional constraints and objective function, into the model.

- Assume level of new product (decision variable) is a non-basic variable

- The only resulting change in the dual problem is to add a new constraint (see Table 6.3).
Introduce a new product or not?

Maximize \[ Z = 3x_1 + 5x_2 + 4x_{\text{new}}, \]
subject to
\[ x_1 + 2x_{\text{new}} \leq 4, \]
\[ 2x_2 + 3x_{\text{new}} \leq 12, \]
\[ 3x_1 + 2x_2 + x_{\text{new}} \leq 18. \]
and
\[ x_1 \geq 0, \quad x_2 \geq 0, \quad x_{\text{new}} \geq 0. \]
Check Complementary basic solution for the dual

- Original optimal solution: \((x_1, x_2, x_3, x_4, x_5) = (2, 6, 2, 0, 0)\).
- Is this solution, along with \(x_{\text{new}} = 0\), still optimal?

- Check Complementary basic solution for the dual

\[ (y_1, y_2, y_3, z_1 - c_1, z_2 - c_2) = \left(0, \frac{3}{2}, 1, 0, 0\right), y_6 > 0 \]

- Results in a new constraint in the dual and a new variable \(y_6\) which if positive just means we need to check the new dual constraint only.
### TABLE 6.1 Primal and dual problems for the Wyndor Glass Co. example

**Primal Problem in Algebraic Form**

Maximize  \( Z = 3x_1 + 5x_2 + 4x_{\text{new}} \),

subject to

\[
\begin{align*}
    x_1 + 2x_{\text{new}} & \leq 4 \\
    2x_2 + 3x_{\text{new}} & \leq 12 \\
    3x_1 + 2x_2 + x_{\text{new}} & \leq 18
\end{align*}
\]

and

\[
\begin{align*}
x_1 & \geq 0, \\
x_2 & \geq 0, \\
x_{\text{new}} & \geq 0.
\end{align*}
\]

**Dual Problem in Algebraic Form**

Minimize  \( W = 4y_1 + 12y_2 + 18y_3 \),

subject to

\[
\begin{align*}
y_1 + 3y_3 & \geq 3 \\
2y_2 + 2y_3 & \geq 5 \\
2y_1 + 3y_2 + y_3 & \geq 4
\end{align*}
\]

and

\[
\begin{align*}
y_1 & \geq 0, \\
y_2 & \geq 0, \\
y_3 & \geq 0.
\end{align*}
\]

Given that \( y_1=0 \), \( y_4=0 \) and \( y_5=0 \) in the dual space

solve for \( y_2, y_3, y_6 \)

\[
\begin{align*}
y_2 &= \frac{3}{2}; \\
y_3 &= 1; \\
y_6 &= \frac{3}{2}
\end{align*}
\]

Complementary Dual BS \((0, 0, 3/2, 0, 0, 3/2, 1)\) to

Primal BS \((2, 6, 0, 2, 0, 0)\); \( W = (0 \times 4 + 3/2 \times 12 + 1 \times 18) = 42 \)
Do not add new product

\[ 2y_1 + 3y_2 + y_3 \geq 4 \]

Plugging in this solution, we see that

\[ 2(0) + 3\left(\frac{3}{2}\right) + (1) \geq 4 \]

is satisfied, so this dual solution is still feasible (and thus still optimal). Consequently, the original primal solution \((2, 6, 0, 2, 0, 0)\), along with \(x_{\text{new} 0}\), is still optimal, so this third possible new product should not be added to the product line.
Dual Simplex

• This algorithm operates on the primal problem exactly as if the simplex method were being applied simultaneously to the dual problem, which can be done because of this property.
  – Because the roles of row 0 and the right side in the simplex tableau have been reversed, the dual simplex method requires that row 0 begin and remain nonnegative while the right side begins with some negative values (subsequent iterations strive to reach a nonnegative right side).
  – Consequently, this algorithm occasionally is used because it is more convenient to set up the initial tableau in this form than in the form required by the simplex method.

• Furthermore, it frequently is used for reoptimization because changes in the original model lead to the revised final tableau fitting this form. This situation is common for certain types of sensitivity analysis, as you will see later in the chapter.
Dual Simplex Algorithm

- In “primal” simplex, RHS column is always non-negative, hence basic solution is feasible at every iteration.
- What if some elements of the RHS column are negative?
- In such a case, primal is infeasible.
- Dual Simplex Algorithm (DSA) addresses such a scenario.
- DSA: particularly useful for re-optimizing a problem after a constraint has been added or some problem parameter has been changed (sensitivity analysis), such that a previously optimal basis is no longer feasible.
Dual Simplex Algorithm: Concept

- **At each iteration of “primal” simplex:**
  - Always maintain primal feasibility (RHS ≥ 0)
  - Drive towards primal optimality (in other words dual feasibility), i.e., coefficients of variables in (-z) row ≤ 0
  - Corresponding dual is always infeasible

- **At each iteration of “dual” simplex**
  - Always maintain primal optimality, i.e., coefficients of variables in (-z) row ≤ 0 [passes optimality test → basic feasible soln in dual space]
  - In other words, always maintain dual feasibility
  - Drive towards primal feasibility (RHS ≥ 0)
  - Terminate when primal feasibility is attained, i.e., all elements in RHS column ≥ 0

- **Dual simplex algorithm tends to be more efficient**
Dual Canonical Form

- All decision variables \( \geq 0 \)
- All RHS coefficients negative (only difference with “primal” simplex)
- All constraints, except non-negativity stated as equalities
- Isolate one decision variable from each constraint with +1 coefficient, which does not appear in any other constraint and appears with a zero coefficient in the objective function
Procedure of Dual Simplex method

- Convert any functional constraint in \( \geq \) form to \( \leq \) form by multiplying both sides by \(-1\)
- Introduce slack variables as needed
- Identify leaving variable
  - variable to leave is the basic variable associated with the constraint with most negative RHS value
- Row corresponding to leaving variable called “pivot row”
- Perform ratio test to identify entering variable
  - Pick all negative coefficients in pivot row \(a_{ij}\)
  - Let \(x_i\) be leaving variable
  - Compute the ratios \(c_j / a_{ij}\) where all \(a_{ij} < 0\)
- Column(variable) that gives smallest ratio enters basis
- Identify pivot element (as in “primal” simplex)
- Divide it by itself to make it 1
- Make other elements in the column of the pivot element = 0 by performing row operations
- Continue till all elements in RHS column become \(\geq 0\)
Apply DSA to the dual of the Wyndor

<table>
<thead>
<tr>
<th>TABLE 6.1 Primal and dual problems for the Wyndor Glass Co. example</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Primal Problem</strong></td>
</tr>
<tr>
<td>in Algebraic Form</td>
</tr>
<tr>
<td>Maximize $Z = 3x_1 + 5x_2,$</td>
</tr>
<tr>
<td>subject to $x_1 \leq 4$</td>
</tr>
<tr>
<td>$2x_2 \leq 12$</td>
</tr>
<tr>
<td>$3x_1 + 2x_2 \leq 18$</td>
</tr>
<tr>
<td>and $x_1 \geq 0, \quad x_2 \geq 0.$</td>
</tr>
</tbody>
</table>

Solve the Dual problem
Convert to problem to Maximization to get the all row zero coefficients $\leq 0$
<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basic Variable</th>
<th>Eq.</th>
<th>Coefficient of:</th>
<th>Right Side</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
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<td>x₁</td>
</tr>
<tr>
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<td>Z</td>
<td>(0)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>x₃</td>
<td>(1)</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>x₄</td>
<td>(2)</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>x₅</td>
<td>(3)</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Z</td>
<td>(0)</td>
<td>1</td>
<td>−3</td>
</tr>
<tr>
<td></td>
<td>x₃</td>
<td>(1)</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>x₂</td>
<td>(2)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>x₅</td>
<td>(3)</td>
<td>0</td>
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<td>2</td>
<td>Z</td>
<td>(0)</td>
<td>1</td>
<td>0</td>
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<td>0</td>
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<td></td>
<td>x₁</td>
<td>(3)</td>
<td>0</td>
<td>1</td>
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</tbody>
</table>
### TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basic Variable</th>
<th>Eq.</th>
<th>Coefficient of:</th>
<th>Right Side</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Z x1 x2 x3 x4 x5</td>
<td></td>
</tr>
<tr>
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<td>Z</td>
<td>(0)</td>
<td>1 -3 -5 0 0 0 0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>x^3</td>
<td>(1)</td>
<td>0 1 0 1 0 0 0 4</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>x^4</td>
<td>(2)</td>
<td>0 2 0 1 0 0 12</td>
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</tr>
<tr>
<td></td>
<td>x^5</td>
<td>(3)</td>
<td>3 2 0 0 0 1 18</td>
<td>18</td>
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<td>(0)</td>
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</tr>
<tr>
<td></td>
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<td>(2)</td>
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</tr>
<tr>
<td></td>
<td>x^5</td>
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</tr>
<tr>
<td>2</td>
<td>Z</td>
<td>(0)</td>
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<tr>
<td></td>
<td>x^1</td>
<td>(3)</td>
<td>1 0 0 0 1 2 1 2</td>
<td>2</td>
</tr>
</tbody>
</table>

### TABLE 7.1 Dual simplex method applied to the Wyndor Glass Co. dual problem

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Basic Variable</th>
<th>Eq.</th>
<th>Coefficient of:</th>
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<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Z y_1 y_2 y_3 y_4 y_5</td>
</tr>
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<td>Z</td>
<td>(0)</td>
<td>1 4 12 18 0</td>
</tr>
<tr>
<td></td>
<td>y_4</td>
<td>(1)</td>
<td>0 -1 0 -3 1 0</td>
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<td></td>
<td>y_5</td>
<td>(2)</td>
<td>0 0 -2 -2 0 1</td>
</tr>
<tr>
<td>1</td>
<td>Z</td>
<td>(0)</td>
<td>1 4 0 6 0 0 6</td>
</tr>
<tr>
<td></td>
<td>y_4</td>
<td>(1)</td>
<td>0 -1 0 -3 1 0</td>
</tr>
<tr>
<td></td>
<td>y_2</td>
<td>(2)</td>
<td>0 0 1 1 0 1 2</td>
</tr>
<tr>
<td>2</td>
<td>Z</td>
<td>(0)</td>
<td>1 2 0 0 2 6</td>
</tr>
<tr>
<td></td>
<td>y_3</td>
<td>(1)</td>
<td>0 0 1 0 0 1 3 0 0 1</td>
</tr>
<tr>
<td></td>
<td>y_2</td>
<td>(2)</td>
<td>0 0 1 0 0 1 3 2</td>
</tr>
</tbody>
</table>

TIM 206 (30155) Introduction to
Example 1

• Consider the following LP

Max \(-3x_1 - 4x_2\)

s.t. \(-2x_1 + x_2 \leq -2\)

\(x_1 + 2x_2 \geq 4\)

\(x_1, x_2 \geq 0\)

Step 1: Multiply second constraint by \(-1\) to convert to \(\leq\) Form

Max \(-3x_1 - 4x_2\)

s.t. \(-2x_1 + x_2 \leq -2\)

\(-x_1 - 2x_2 \leq -4\)

\(x_1, x_2 \geq 0\)
Example 1-Contd...

Step 2: Add slack variables, convert into dual canonical form

Max \(-3x_1 - 4x_2\)

s.t. \(-2x_1 + x_2 + x_3 = -2\)
\(-x_1 - 2x_2 + x_4 = -4\)

\(x_1, x_2, x_3, x_4 \geq 0\)

Canonical form shown below

\(-2x_1 + x_2 + x_3 + 0x_4 = -2\)
\(-x_1 - 2x_2 + 0x_3 + x_4 = -4\)
\(-3x_1 - 4x_2 + 0x_3 + 0x_4 = 0 \) [Row 0]
### 1st Tableau

<table>
<thead>
<tr>
<th>Basic Vars</th>
<th>RHS</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_3$</td>
<td>-2</td>
<td>-2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$x_4$</td>
<td>-4</td>
<td>-1</td>
<td>-2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(-z)</td>
<td>0</td>
<td>-3</td>
<td>-4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Ratio</td>
<td>-3/-1=3</td>
<td>-4/-2=2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- **Pivot Row**: make it 1 and other elements in column of $x_2 = 0$ by row operations.
- **Leaves**: with most negative RHS value.
- **Pivot Element**: smallest ratio enters basis.
<table>
<thead>
<tr>
<th>Basic Vars</th>
<th>RHS</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_3$</td>
<td>-4</td>
<td>-5/2</td>
<td>0</td>
<td>1</td>
<td>½</td>
</tr>
<tr>
<td>$x_2$</td>
<td>2</td>
<td>½</td>
<td>1</td>
<td>0</td>
<td>-½</td>
</tr>
<tr>
<td>(-z)</td>
<td>8</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>Ratio</td>
<td>-1/(-5/2)=2/5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$x_3$ leaves, $x_1$ enters
### 3rd Tableau

<table>
<thead>
<tr>
<th>Basic Vars</th>
<th>RHS</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>8/5</td>
<td>1</td>
<td>0</td>
<td>-2/5</td>
<td>-1/5</td>
</tr>
<tr>
<td>$x_2$</td>
<td>6/5</td>
<td>0</td>
<td>1</td>
<td>1/5</td>
<td>-2/5</td>
</tr>
<tr>
<td>(-z)</td>
<td>48/5</td>
<td>0</td>
<td>0</td>
<td>-2/5</td>
<td>-11/5</td>
</tr>
<tr>
<td>Ratio</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Note:**

All RHS elements are now $\geq 0$

Hence we are done

Optimal solution: $z=-48/5$, $x_1=8/5$, $x_2=6/5$
Points to Note

• In “primal” simplex, first identify entering variable, then leaving variable
• In “dual” simplex, first identify leaving variable, then entering variable
• At each iteration, all elements of (-z) row ≤ 0
• At each iteration, the dual to the original problem is always feasible
  – Verify this by writing the dual to the original problem
  – Obtain values form dual multipliers from each tableau
    • At each iteration, dual multipliers = values of slacks in (z) row, e.g. at 2nd iteration, dual multipliers are 0 and 2
Example 2

Max $-x_1 - 2x_2$

s.t.

$-x_1 + 2x_2 - x_3 \leq -2$

$-2x_1 - x_2 + x_3 \leq -6$

$x_1, x_2, x_3 \geq 0$
### 1\text{st} Tableau

<table>
<thead>
<tr>
<th>Basic Vars</th>
<th>RHS</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_4$</td>
<td>-4</td>
<td>-1</td>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$x_5$</td>
<td>-6</td>
<td>-2</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(-z)</td>
<td>0</td>
<td>-1</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Ratio</td>
<td></td>
<td>$-1/-2=1/2$</td>
<td>$-2/-1=2$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$x_5$ leaves, $x_1$ enters
### 2nd Tableau

<table>
<thead>
<tr>
<th>Basic Vars</th>
<th>RHS</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_4$</td>
<td>-1</td>
<td>0</td>
<td>5/2</td>
<td>-3/2</td>
<td>1</td>
<td>-1/2</td>
</tr>
<tr>
<td>$x_1$</td>
<td>3</td>
<td>1</td>
<td>1/2</td>
<td>-1/2</td>
<td>0</td>
<td>-1/2</td>
</tr>
<tr>
<td>(-z)</td>
<td>3</td>
<td>0</td>
<td>3/2</td>
<td>-1/2</td>
<td>0</td>
<td>-1/2</td>
</tr>
<tr>
<td>Ratio</td>
<td></td>
<td></td>
<td></td>
<td>(-1/2)/(3/2)=1/3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**$x_4$ leaves, $x_3$ enters**
### 3rd Tableau

<table>
<thead>
<tr>
<th>Basic Vars</th>
<th>RHS</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_3$</td>
<td>2/3</td>
<td>0</td>
<td>-5/3</td>
<td>1</td>
<td>-2/3</td>
<td>1/3</td>
</tr>
<tr>
<td>$x_1$</td>
<td>10/3</td>
<td>1</td>
<td>-1/3</td>
<td>0</td>
<td>-1/3</td>
<td>-1/3</td>
</tr>
<tr>
<td>(-z)</td>
<td>10/3</td>
<td>0</td>
<td>-7/3</td>
<td>0</td>
<td>-1/3</td>
<td>-1/3</td>
</tr>
<tr>
<td>Ratio</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Optimal solution obtained:** $z = -10/3$, $x_1 = 10/3$, $x_2 = 0$
Example of DSA; and Restart after RHS change

• http://www.me.utexas.edu/~jensen/ORMM/supplements/methods/lpmethod/S1_dualsimplex.pdf

• DSA example

• Restarting after Changing the Right-Hand-Side Constants
Transportation and Network Flow problems

• Transportation problem [Transportation Simplex Method]
  – Received this name because many of its applications involve determining how to optimally transport goods. However, some of its important applications (e.g., production scheduling) actually have nothing to do with transportation.

• Assignment problem,
  – involves such applications as assigning people to tasks. Although its applications appear to be quite different from those for the transportation problem, we shall see that the assignment problem can be viewed as a special type of transportation problem.

• Linear programming problems involving networks, including the minimum cost flow problem
  – [Network Simplex Method, Transportation and Assignment are special cases of the minimum cost flow problem]
Sparse Matrix: Exploit structure $\rightarrow$ new algo

- Applications of the transportation and assignment problems tend to require a very large number of constraints and variables, so a straightforward computer application of the simplex method may require an exorbitant computational effort.

- Fortunately, a key characteristic of these problems is that most of the $a_{ij}$ coefficients in the constraints are zeros, and the relatively few nonzero coefficients appear in a distinctive pattern. As a result, it has been possible to develop special streamlined algorithms that achieve dramatic computational savings by exploiting this special structure of the problem.
Transportation Simplex Method

Major Characteristics:
• Several sources (plant) with given supply capacity.
• Several destinations (warehouse) with given demand.
• Cost Matrix: unit transportation cost from each source to each destination.

Assumptions:
• No transshipment between any two destinations/two sources.
• All demand/capacity constraints should be attempted to meet.

Decisions: How much should be shipped from each source to each destination so that total transportation cost is minimized?

Why Develop a New Method? (LP can certainly be applied)
• LP is still NP-complete to large size problems.
• Computational efficient methods are desirable to large size problems.

Transportation Problem Model: Tabular Presentation
<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Output</th>
</tr>
</thead>
</table>
| Cannery
| 1   | 464  | 513  | 654  | 867   | 75     |
| 2   | 352  | 416  | 690  | 791  | 125    |
| 3   | 995  | 682  | 388  | 685  | 100    |
| Allocation  | 80   | 65   | 70   | 85   |        |
Network representation of the P & T Co. problem.
1. Decision Variable:
   Since we have to determine how much canned peas shipments is sent from each plant to each city;

\[ X_{ij} = \text{Amount of canned peas produced at plant } i \text{ and sent to city } j \]

\[ X_{14} = \text{Amount of canned peas produced at plant } 1 \text{ and sent to city } 4 \]
Minimize Shipping Costs

Minimize \[ Z = 464x_{11} + 513x_{12} + 654x_{13} + 867x_{14} + 352x_{21} + 416x_{22} + 690x_{23} + 791x_{24} + 995x_{31} + 682x_{32} + 388x_{33} + 685x_{34}, \]

subject to the constraints

\[
\begin{align*}
    x_{11} + x_{12} + x_{13} + x_{14} &= 75 \\
    x_{21} + x_{22} + x_{23} + x_{24} &= 125 \\
    x_{31} + x_{32} + x_{33} + x_{34} &= 100 \\
    x_{11} + x_{21} + x_{31} &= 80 \\
    x_{12} + x_{22} + x_{32} &= 65 \\
    x_{13} + x_{23} + x_{33} &= 70 \\
    x_{14} + x_{24} + x_{34} &= 85 \\
\end{align*}
\]

and

\[ x_{ij} \geq 0 \quad (i = 1, 2, 3; j = 1, 2, 3, 4). \]

<table>
<thead>
<tr>
<th>Cannery</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>464</td>
<td>513</td>
<td>654</td>
<td>867</td>
<td>75</td>
</tr>
<tr>
<td>2</td>
<td>352</td>
<td>416</td>
<td>690</td>
<td>791</td>
<td>125</td>
</tr>
<tr>
<td>3</td>
<td>995</td>
<td>682</td>
<td>388</td>
<td>685</td>
<td>100</td>
</tr>
<tr>
<td>Allocation</td>
<td>80</td>
<td>65</td>
<td>70</td>
<td>85</td>
<td></td>
</tr>
</tbody>
</table>
The feasible solutions property: A transportation problem will have feasible solutions if and only if

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j.$$
Transportation Problem
## Transportation Cost

<table>
<thead>
<tr>
<th>From (Sources)</th>
<th>To (Destinations)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Albuquerque</td>
<td>Boston</td>
</tr>
<tr>
<td>DesMoines</td>
<td>$5</td>
<td>$4</td>
</tr>
<tr>
<td>Evansville</td>
<td>$8</td>
<td>$4</td>
</tr>
<tr>
<td>Fort Lauderdale</td>
<td>$9</td>
<td>$7</td>
</tr>
</tbody>
</table>
### Transportation Table

<table>
<thead>
<tr>
<th></th>
<th>Albuquerque (A)</th>
<th>Boston (B)</th>
<th>Cleveland (C)</th>
<th>Factory Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Des Moines (D)</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>100</td>
</tr>
<tr>
<td>Evansville (E)</td>
<td>8</td>
<td>4</td>
<td>3</td>
<td>300</td>
</tr>
<tr>
<td>Fort Lauderdale (F)</td>
<td>9</td>
<td>7</td>
<td>5</td>
<td>300</td>
</tr>
<tr>
<td>Warehouse Requirements (Demand)</td>
<td>300</td>
<td>200</td>
<td>200</td>
<td>(700)</td>
</tr>
</tbody>
</table>
Steps in Solving the Transportation Problem

1. Arrange Problem in Table Form
2. Balance Table (if necessary)
3. Find an Initial Feasible Solution
4. Is Solution Optimal?
   - Yes: Problem Is Solved
   - No: Generate Improved Solution

---
Solution Procedure

1. Balancing Table: Make $\sum d_i = \sum S_i$
   - add a "Dummy" source when $\sum d_i > \sum S_i$, or
   - add a "Dummy" destination when $\sum S_i > \sum d_i$.

2. Finding an initial solution: (several methods)
   - Upper-Left Corner (Northwest) Method
   - Least-Cost (Large-Profit) Method
   - VAM Method

3. Testing Optimality: (Can current solution be improved?)
   - Stepping-stone Procedure
   - Modified-Distribution Method

4. Improving current solution toward a better solution.

Special cases in Transportation Problem:
   - Prohibited transportation route (take large "M" as cost)
   - Multiple optimal solution is possible.
   - Transshipment allowed/Route shipment capacity limits
     (All special cases can be easily formulated and solved by LP.)

Application Examples:
   - Production Planning/Scheduling/Routing Selection/....
**Transportation Problem**

**Maximization Objectives**

**Problem Description**
Klein Chemicals, Inc. manufactures a product at two plants (Clifton Springs and Danville) and ships it to four different customers (denoted D₁, D₂, D₃ and D₄)

Profit per unit for shipping from each plant to each customer.

**Profit Per Unit:**

<table>
<thead>
<tr>
<th>Plant</th>
<th>Clifton Springs</th>
<th>$32</th>
<th>$34</th>
<th>$32</th>
<th>$40</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Danville</td>
<td>$34</td>
<td>430</td>
<td>$28</td>
<td>$38</td>
</tr>
</tbody>
</table>

**Customers**

<table>
<thead>
<tr>
<th>D₁</th>
<th>D₂</th>
<th>D₃</th>
<th>D₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>Profit per unit</td>
<td>$32</td>
<td>$34</td>
<td>$32</td>
</tr>
</tbody>
</table>
### Plant Capacities and Customer Orders:

<table>
<thead>
<tr>
<th>Plant</th>
<th>Capacities (Units)</th>
<th>Customers</th>
<th>Orders (Units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clifton</td>
<td>5000</td>
<td>$D_1$</td>
<td>2000</td>
</tr>
<tr>
<td>Danville</td>
<td>3000</td>
<td>$D_2$</td>
<td>5000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$D_3$</td>
<td>3000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$D_4$</td>
<td>2000</td>
</tr>
<tr>
<td>Total Capacity</td>
<td>8000</td>
<td>Total Orders</td>
<td>12000</td>
</tr>
</tbody>
</table>
Since the number of units ordered exceeds the plant capacities, we must add a dummy plant with a capacity of 4000 units.

The Complete Transportation Tableau is:

<table>
<thead>
<tr>
<th>From Plant</th>
<th>To Customer</th>
<th>Plant Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$D_1$</td>
<td>$D_2$</td>
</tr>
<tr>
<td>Clifton Springs</td>
<td>32</td>
<td>34</td>
</tr>
<tr>
<td>Danville</td>
<td>34</td>
<td>30</td>
</tr>
<tr>
<td>Dummy</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Customer Demand</td>
<td>2000</td>
<td>5000</td>
</tr>
</tbody>
</table>
Initial Feasible Solution—Min Cost Method:
<table>
<thead>
<tr>
<th></th>
<th>2000</th>
<th>5000</th>
<th>3000</th>
<th>2000</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>32</td>
<td>34</td>
<td>32</td>
<td>40</td>
</tr>
<tr>
<td>2</td>
<td>34</td>
<td>30</td>
<td>28</td>
<td>38</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>3000</th>
<th>5000</th>
<th>3000</th>
<th>2000</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Apply MODI Method to Find Optimal Solution:

\[
\begin{array}{cccc}
\mu_i & v_j & \mu & \\
0 & 38 & 34 & 34 & 40 \\
-4 & 0 & 3000 & -2 & 2000 \\
-34 & 34 & 1000 & -2 & 2 \\
\end{array}
\]

\[
\begin{array}{cccc}
\mu_i & v_j & \mu & \\
0 & 36 & 34 & 34 & 40 \\
-2 & 0 & 4000 & -2 & 1000 \\
-34 & 34 & 1000 & -2 & 1000 \\
\end{array}
\]
Optimal Solution:

<table>
<thead>
<tr>
<th>Route</th>
<th>Units</th>
<th>Profit</th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clifton Springs to D₂</td>
<td>4000</td>
<td>$34</td>
<td>$136,000</td>
</tr>
<tr>
<td>Clifton Springs to D₄</td>
<td>1000</td>
<td>40</td>
<td>40,000</td>
</tr>
<tr>
<td>Danville to D₁</td>
<td>2000</td>
<td>34</td>
<td>68,000</td>
</tr>
<tr>
<td>Danville to D₄</td>
<td>1000</td>
<td>38</td>
<td>38,000</td>
</tr>
<tr>
<td>Dummy to D₂</td>
<td>1000</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Dummy to D₃</td>
<td>3000</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Total Profit = $282,000

Each plant should produce at its capacity

Customer $D₂$ is not satisfied (1000 units short)
Customer $D₃$ is not satisfied (3000 units short)
The Stepping-Stone Method

1. Select any unused square to evaluate.
2. Begin at this square. Trace a closed path back to the original; square via square that are currently being used (only horizontal or vertical moves allowed).
3. Place + in unused square; alternate – and + on each corner square of the closed path.
4. Calculate improvement index: add together the unit figures found in each square containing a +; subtract the unit cost figure in each square containing a -.
5. Repeat steps 1-4 for each unused square.
6. See video

1. http://www.youtube.com/watch?v=RGKQXBL2YWo
Stepping-Stone Method: Tracing a Closed Path for the Des Moines to Cleveland route

<table>
<thead>
<tr>
<th></th>
<th>Albuquerque (A)</th>
<th>Boston (B)</th>
<th>Cleveland (C)</th>
<th>Factory Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Des Moines (D)</td>
<td>100</td>
<td>4</td>
<td>Start 3</td>
<td>100</td>
</tr>
<tr>
<td>Evansville (E)</td>
<td>200</td>
<td>100</td>
<td>3</td>
<td>300</td>
</tr>
<tr>
<td>Fort Lauderdale (F)</td>
<td>100 +</td>
<td>200</td>
<td>200</td>
<td>300</td>
</tr>
<tr>
<td>Warehouse Requirements</td>
<td>300</td>
<td>200</td>
<td>200</td>
<td>/00</td>
</tr>
</tbody>
</table>
Assignment Problem

Major Characteristics:
- Finite number of Team-Job (or the like) to be assigned.
- Assignment must be made based on one-to-one basis.
- Payoff matrix: cost (profit) for each possible assignment.

Available Solution Techniques to Assignment Problem:
- Linear Programming
- Dynamic Programming
- Integer IP (Branch-and-Bound)
- Complete Enumeration
- Transportation Model (special case with all $S_i = d_i = 1$)
- Hungrain Method (A "smart" method for large size problems)

Special Cases in Assignment Problem:
- Prohibited Assignment (assign a large "M" as cost)
- It is possible to have multiple optimal solutions.
- Special arrangement for a fixed assignment or One-to-More assignment.
Special Solution Procedure

Hungrain Method: (Tabular Form)
- Principle: Add (or subtract) a constant to all cells of a row (or column) in the table will not change final optimal solution.
- Four-Step Procedure
  - For Max Problem: (Transforming into Min Problem)
    - Change the sign ("+" or "-"") of each cell in the table.
    - Add the largest cell value to all cells.
  (Examples)

Summary:
- Both Transportation Problem and Assignment Problem are special cases of LP problem, and can be solved by Simplex method.
- Tabular solution technique mainly provides a more efficient way for some large size practical problems.
- Many practical managerial problems can be formulated and solved by Transportation or Assignment problem techniques.
Medical Supply Transportation Problem

• A Medical Supply company produces catheters in packs at three productions facilities.
• The company ships the packs from the production facilities to four warehouses.
• The packs are distributed directly to hospitals from the warehouses.
• The table on the next slide shows the costs per pack to ship to the four warehouses.
# Medical Supply

<table>
<thead>
<tr>
<th>FROM PLANT</th>
<th>TO WAREHOUSE</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Seattle</td>
</tr>
<tr>
<td>Juarez</td>
<td>$19</td>
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<tr>
<td>Seoul</td>
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<tr>
<td>Tel Aviv</td>
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</tbody>
</table>

**Capacity**
- Juarez: 100
- Seoul: 300
- Tel Aviv: 200

**Demand**
- Seattle: 150
- New York: 100
- Phoenix: 200
- Miami: 150

Source: Adapted from Lapin, 1994
Number of constraints = number of rows + number of columns

Total plant capacity must equal total warehouse demand. Although this may seem unrealistic in real world application, it is possible to construct any transportation problem using this model.

Source: Adapted from Lapin, 1994
Northwest Corner Method

<table>
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</table>

Begin with a blank shipment schedule. Note the shipping costs in the upper right hand corner of each cell.

Source: Adapted from Lapin, 1994
### Northwest Corner Method

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Start in the upper left-hand corner, “northwest corner” of the schedule and place the largest amount of capacity and demand available in that cell. Seattle demands 150 and Jaurez has a capacity of 100.

Source: Adapted from Lapin, 1994
# Northwest Corner Method

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Since Juarez capacity is depleted move down to repeat the process for the Seoul to Seattle cell. Seoul has sufficient capacity but Seattle can only take another 50 packs of demand.
Now move to the next cells to the right and assign capacity for Seoul to warehouse demand until depleted. Then move down to the Tel Aviv row and repeat the process.
The previous slides show the process of satisfying all constraints and allows us to begin with a starting feasible solution. Multiply the quantity in each cell by the cost.
For non empty cells: $c_{ij} = r_i + k_j$

Assign zero as the row number for the first row.

**Please Skip for TIM 206**

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</table>

$r_j = 0$

$k_s = 19$

Source: Adapted from Lapin, 1994

Please Skip for TIM 206
For non empty cells: \( c_{ij} = r_i + k_j \)

Assign zero as the row number for the first row.

Note: Always use the newest \( r \) value to compute the next \( k \).

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\[ 15 = r_s + 19 \]
\[ r_s = -15 + 19 \]
\[ = -4 \]
**Please Skip for TIM 206**

For non empty cells: \( c_{ij} = r_i + k_j \)

Assign zero as the row number for the first row.

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- **Source:** Adapted from Lapin, 1994
- **18 = -4 + k_p**
- **18 + 4 = k_p** = 22
- **k_s = 19**
- **k_p = 22**

Skip cell SN, mark it * for later and move on to cell SP.
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For non-empty cells: \( c_{tp} = r_t + k_p \)

Assign zero as the row number for the first row then use the newest \( r \) value to compute the next \( k \).

15 = \( r_t + 22 \)
15 - 22 = \( r_t \)
= -7

Source: Adapted from Lapin, 1994

Please Skip for TIM 206
For non empty cells: \( c_{ij} = r_i + k_j \)

Assign zero as the row number for the first row then use the newest \( r \) value to compute the next \( k \).

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Source: Adapted from Lapin, 1994

Please Skip for TIM 206

- \( k_s = 19 \) (b)
- \( k_p = 22 \) (d)
- \( k_m = 29 \) (f)
- \( r_j = 0 \) (a)
- \( r_s = -4 \) (c)
- \( r_t = -7 \) (e)
For non empty cells: \( c_{ij} = r_i + k_j \)

Assign zero as the row number for the first row then use the newest \( r \) value to compute the next \( k \).

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Source: Adapted from Lapin, 1994
Next calculate empty cells using: $c_{ij} - r_i - k_j$

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Source: Adapted from Lapin, 1994

**Improvement Difference >>** JN = 7 - 0 - 25 = -18

Please Skip for TIM 206
Next calculate empty cells using: \( c_{ij} - r_i - k_j \)

**Improvement Difference >>**

\[ JP = 3 - 0 - 22 = -19 \]

Please Skip for TIM 206

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Demand: 150, 100, 200, 150, 600

Source: Adapted from Lapin, 1994

\( r_j = 0 \) (a)
\( r_s = -4 \) (c)
\( r_t = -7 \) (e)

\( k_s = 19 \) (b)
\( k_n = 25 \) (g)
\( k_p = 22 \) (d)
\( k_m = 29 \) (f)
Next calculate empty cells using: $c_{ij} - r_i - k_j$

$\text{Improvement Difference >> JM} = 21 - 0 - 29 = -8$

Please Skip for TIM 206

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$k_s = 19$  
$k_n = 25$  
$k_p = 22$  
$k_m = 29$  
$r_j = 0$  
$r_s = -4$  
$r_t = -7$
Next calculate empty cells using: $c_{ij} - r_i - k_j$

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Source: Adapted from Lapin, 1994

Improvement Difference >> $SM = 6 - (-4) - 29 = -19$

Please Skip for TIM 206

$\begin{align*}
  r_j &= 0 \\
  r_s &= -4 \\
  r_t &= -7
\end{align*}$
Next calculate empty cells using: $c_{ij} - r_i - k_j$

**Please Skip for TIM 206**

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**Demand**

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**Capacity**

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**Source:** Adapted from Lapin, 1994

$TS = 11 - (-7) - 19 = -1$

$r_j = 0$

$r_s = -4$

$r_t = -7$

$k_s = 19$

$k_n = 25$

$k_p = 22$

$k_m = 29$
Next calculate empty cells using: $c_{ij} - r_i - k_j$

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$TN = 14 - (-7) - 25 = -4$

$\text{Please Skip for TIM 206}$

Source: Adapted from Lapin, 1994
Next calculate empty cells using: \( c_{ij} - r_i - k_j \)

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<td>From</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>J</td>
<td>100</td>
<td>-18</td>
<td>-19</td>
<td>-8</td>
<td>100</td>
</tr>
<tr>
<td>S</td>
<td>50</td>
<td>100</td>
<td>150</td>
<td>-19</td>
<td>300</td>
</tr>
<tr>
<td>T</td>
<td>11</td>
<td>14</td>
<td>50</td>
<td>150</td>
<td>200</td>
</tr>
<tr>
<td>Demand</td>
<td>150</td>
<td>100</td>
<td>200</td>
<td>150</td>
<td>600</td>
</tr>
</tbody>
</table>

Source: Adapted from Lapin, 1994

Please Skip for TIM 206
Next calculate the entering cell by finding the empty cell with the greatest absolute negative improvement difference. Cells JP and SM are tied for the greatest improvement at $19 per pack. Break the tie and arbitrarily choose JP. JP becomes the entering cell. Place a + sign in cell JP.

<table>
<thead>
<tr>
<th>From</th>
<th>S</th>
<th>N</th>
<th>P</th>
<th>M</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>J</td>
<td>100</td>
<td>-18</td>
<td>100</td>
<td>-8</td>
<td>100</td>
</tr>
<tr>
<td>S</td>
<td>50</td>
<td>100</td>
<td>150</td>
<td>-19</td>
<td>300</td>
</tr>
<tr>
<td>T</td>
<td>-1</td>
<td>-4</td>
<td>50</td>
<td>150</td>
<td>200</td>
</tr>
<tr>
<td>Demand</td>
<td>150</td>
<td>100</td>
<td>200</td>
<td>150</td>
<td>600</td>
</tr>
</tbody>
</table>

Source: Adapted from Lapin, 1994

Note: Except for the entering cell all changes must involve nonempty cells.
Continue around the closed loop until all tradeoffs are completed.

Previous cost was $11,500 and the new is:

<table>
<thead>
<tr>
<th></th>
<th>S</th>
<th>N</th>
<th>P</th>
<th>M</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>From</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| J     |  -18 | -19 | 100 | -8 | 100 | $r_j = 0$
| S     |  150 | 100 |  50 | 18 | 300 | $r_s = -4$
| T     |   |   |  50 | 150 | 200 | $r_t = -7$

Demand

\[ k_s = 19 \quad k_n = 25 \quad k_p = 22 \quad k_m = 29 \]

Note: Except for the entering cell all changes must involve nonempty cells.
Please Skip for TIM 206

Note: The r and k values and the improvement difference values have changed.
Please Skip for TIM 206

Previous cost was $9,600, now the new is:

<table>
<thead>
<tr>
<th></th>
<th>S</th>
<th>N</th>
<th>P</th>
<th>M</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>J</td>
<td>19</td>
<td>17</td>
<td>100</td>
<td>21</td>
<td>100</td>
</tr>
<tr>
<td>S</td>
<td>150</td>
<td>100</td>
<td>50</td>
<td>6</td>
<td>300</td>
</tr>
<tr>
<td>T</td>
<td>11</td>
<td>14</td>
<td>100</td>
<td>22</td>
<td>200</td>
</tr>
</tbody>
</table>

Demand: 150 100 200 150 600

k_s = 0  k_n = 6  k_p = 3  k_m = 10

r_j = 0  r_s = 15  r_t = 12

C = $8,650

Note: The r and k values and the improvement difference values have changed.
Please Skip for TIM 206

Begin another iteration choosing the empty cell with the greatest absolute negative improvement difference.

Previous cost was $8,650, now the new is:

$$C = 6,350$$

Note: The r and k values and the improvement difference values have changed.
Please Skip for TIM 206

$6,350

Begin another iteration choosing the empty cell with the greatest absolute negative improvement difference.

<table>
<thead>
<tr>
<th>From</th>
<th>S</th>
<th>N</th>
<th>P</th>
<th>M</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>J</td>
<td>19</td>
<td>17</td>
<td>100</td>
<td>21</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>5</td>
<td>150</td>
<td>31</td>
<td></td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>21</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>14</td>
<td>150</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td></td>
<td>20</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: The $r$ and $k$ values and the improvement difference values have changed.
In five iterations the shipping cost has moved from $11,500 to $6,250. There are no remaining empty cells with a negative value.

Note: The r and k values and the improvement difference values have changed.
Case Studies in Digital Advertising
Advertising

• Advertising is a paid, one-way communication
  1. Deliver marketing messages and attract new customers
  2. To inform potential customers about products and services and how to obtain and use them.
  3. Branding → Direct action
    • Many advertisements are also designed to generate increased consumption of those products and services through the creation and reinforcement of brand image and brand loyalty (ads contain both factual information and persuasive messages).
  4. Use every major medium
    • To deliver these messages, including: television, radio, movies, magazines, newspapers, video games, the Internet, and billboards.
Digital Advertising

• Online advertising is a form of advertising utilizing the Internet and World Wide Web in order to deliver marketing messages and attract customers [wikipedia.com]

• Advertising annoys people! Advertising works!
  – "Half the money I spend on advertising is wasted; the trouble is, I don't know which half." - John Wanamaker, father of modern advertising. [Credit assignment]
  – "I do not regard advertising as entertainment or an art form, but as a medium of information...“, “Ogilvy on Advertising” by David Ogilvy

• Goals of Online advertising

  A – Deliver/push an advertiser’s message with quantifiable measures of consumer interest

  A+P – Generate ROI for the advertiser and revenue for the publisher

  P+C – Enable ads as a medium of information (true in the case of search)
Advertising makes up ~2% of US GDP

"Half the money I spend on advertising is wasted; the trouble is, I don't know which half." - John Wanamaker, father of modern advertising.

Less than 1% of all impressions lead to measurable ROI

Despite its problems (Attribution etc.)

• US GDP = $14.1 Trillion (Global $56 Trillion, $56 \times 10^{12})

• US Advertising Spend
  – ~$275 Billion across all media
    • (2% of GDP since the early 1900s)

• In 2008, Worldwide online advertising was $65B
  – I.e., about 10% of all ad spending across all media [IDC, 2008]
  – $23 Billion in US; $2 Billion in China; $2 Billion in Latin America;
  – $20B (Europe); and Russia accounted for $720 million
Purchase Funnel

Deliver marketing messages and attract customers and sell products/services

**Awareness**

**Consideration**

**Conversion**

- Targeting Demo, Geo and Content related websites
- SEM: User who is searching for your product
- Site Retargeting: Users who previously visited your website
- Users who are on your website
What marketers want via advertising?

- Deliver marketing messages and attract customers and sell products/services

<table>
<thead>
<tr>
<th>Goal</th>
<th>Activity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduce:Reach</td>
<td>Media Planning</td>
</tr>
<tr>
<td>Influence:Brand</td>
<td>Ad Effectiveness (CTR, site visits)</td>
</tr>
<tr>
<td>Close</td>
<td>Marketing Effectiveness (Transactions, ACR, Credit Assignment)</td>
</tr>
<tr>
<td>Network Effect</td>
<td>Referrals/Advocacy</td>
</tr>
</tbody>
</table>
Media Channels for Advertising

• Advertising online comes in all shapes and sizes and we run into it all the time be it through
  – Websearch
  – Reading the newspaper online
  – Paying the bills
  – Listening to music
  – Watching a video
  – Purchasing a book
  – Mobile device-based apps (phones, Tablet computers)
Sponsored Search

AdSponsored Search

Website Advertising
Display Your Ad for Free & Pay Only
When Customers Respond to Your Ad!
adswords.google.com

Marketing & Advertising
Web, Print, Lead Generation, SEO Technology & Startup Specialists
www.glassCompany.com
San Francisco-Oakland-San Jose, CA

Free Online Advertising
Get Listed on Major Search Engines with a 30 Day Free Trial! No Risk!
www.Yodle.com

Facebook Advertising
Reach the exact audience you want with relevant targeted ads.
www.facebook.com/ads/

Advertise in Your Area
Attract Local Customers to Your Business With Direct Advertising!
www.valpak.com/advertise
San Francisco-Oakland-San Jose, CA

MySpace Advertising
Target ads by over 1,100 hobbies & interests. Budgets as low as $5/day
Advertise.MySpace.com

Buttons, Stickers & More
Custom Creations with your Message Service & Quality you can Count on!
www.ELBlueCreations.com
Local Search @ AT&T Interactive

Organic Business Listings

North Sponsored

Sponsored Listing 1

Sponsored Listing 2

Organic Listing 1

Organic Listing 2

Sponsored San Francisco Results

Sushi Rock Inc.
1000 Polk St, San Francisco, CA 94109
(415) 345-1690

Sanraku
704 Sutter St, San Francisco, CA 94109
(415) 771-0003

Sushi Bistro
445 Baboe St, San Francisco, CA 94118
(415) 504-1472

Zushi-Puzzle Inc
1910 Lombard St, San Francisco, CA 94110
(415) 666-2071

Izumi Ya Restaurant
1581 Webster St, San Francisco, CA 94115
(415) 441-6687

ANZU
222 Mason St, San Francisco, CA 94102
(415) 418-7918

Sushi Bistro
Display Ads

- Also called banner ads
- Served by web sites
- Similar to ‘Display’ or ‘Hoardings’ on road side
- Ads are targeted based on the demographics of the visitors of the web site
Contextual Ads:
Target based on text of page

- Served by the web site to its visitors
- Ad network selects ads that are highly related to the content of the web page
Contextual Advertising on Webpages

For standards see IAB
http://www.iab.net/standards/adunits.asp
House Ads at AMEX
OA is cavalier! : business models; ad placement; e:b wants to be online
Advertising: a supply-demand marketplace

**DEMAND**
Advertiser wishes to reach consumers

**SUPPLY**
Ad Slots for sale

---

**Formal Relationship**

Advertiser → Agency → Publisher → Consumers

---

Advertisements
Supply

DEMAND
Advertiser wishes to reach consumers

Ads

Ad
Agency

SUPPLY

• Sponsored Search
• Contextual
• Display
• Classified
• Email
• Social
• Mobile

Consumers
From Mad Men To Wall Street and beyond!

<table>
<thead>
<tr>
<th>Banner</th>
<th>Click+Data</th>
<th>Personal</th>
<th>Social</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Set in New York City, <em>Mad Men</em> begins in 1960 at the fictional Sterling Cooper advertising agency on New York City's Madison Avenue.</td>
<td>Increasingly</td>
<td>2007</td>
<td>Data</td>
</tr>
<tr>
<td>Human Intensive</td>
<td>Technology</td>
<td>Personalization</td>
<td></td>
</tr>
<tr>
<td>Lots of guess work</td>
<td>Data Driven</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Forward Market</td>
<td>Forward Market</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Spot Markets</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1st Generation

Advertisers still in broadcast mode

2nd Generation

3rd Generation

YoY: Double digit growth
### Ad Network Architecture: Forward Market

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>Ad1</th>
<th>Ad2</th>
<th>(Ad_j)</th>
<th>(Ad_m)</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>PageViews</td>
</tr>
<tr>
<td><strong>PubZone 1</strong></td>
<td></td>
<td>(d_{ij})</td>
<td>(d_{ij})</td>
<td>(d_{ij})</td>
<td>(d_{ij})</td>
<td>35</td>
</tr>
<tr>
<td><strong>PubZone 2</strong></td>
<td></td>
<td>…</td>
<td>…</td>
<td>…</td>
<td>…</td>
<td>50</td>
</tr>
<tr>
<td><strong>PubZone 3</strong></td>
<td></td>
<td>(d_{ij})</td>
<td>(d_{ij})</td>
<td>(d_{ij})</td>
<td>(d_{ij})</td>
<td>15</td>
</tr>
<tr>
<td><strong>Demand Contracted PageViews</strong></td>
<td>45</td>
<td>20</td>
<td>30</td>
<td>5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Ad Network Flow:**
- **From:** Publishers
- **To:** Advertisers
- **Supply:** Page Views

**Key Elements:**
- **Crawl:** (Index, Features)
- **ML:** AB Test, DashBoard
- **Ad Network:** From Advertisers to Publishers
- **Yield Management:** Ad Network Users
- **Analytics:** ADashBoard, PDashBoard
- **Creatives:** Ad upload/SelfServe
- **DashBoard:** WebPage, SERP, WWW

**WebPage Structure:**
- **PubZone:** 1, 2, 3
- **Demand:** Contracted Page Views
Guaranteed Delivery (Futures Market)

- Main goal of advertisers: Brand Awareness
- Revenue Model: Cost per impression (CPM)
- Traditional Advertising Model:
  1. Ads are targeted to users based on demographics and other behavioral features
     1. GM ads on Finance shown to “males above 55”
     2. Mortgage ad shown to “everybody on Facebook logout page”
  2. Book a slot well in advance
     - “2M impressions in Jan next year”
     - Future impressions must be **guaranteed** by the ad network
     - Prices are significantly higher than in Spot market for display (aka Non-guaranteed or NGD)
Maximize Revenue: Ad Allocation Example

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>Supply PageViews</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ad1</td>
<td>Ad2</td>
</tr>
<tr>
<td>PubZone 1</td>
<td>d_{ij}</td>
<td>d_{ij}</td>
</tr>
<tr>
<td>PubZone 2</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>PubZone 3</td>
<td>d_{ij}</td>
<td>d_{ij}</td>
</tr>
<tr>
<td>Demand Contracted</td>
<td>45</td>
<td>20</td>
</tr>
<tr>
<td>PageViews</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Use LP to generate the Ad display schedule to maximize my revenue (or rev proxy, i.e., CTR)
Ad Networks and Optimisation

- Allocation of Ads to Publisher real estate
  - Give ads play in network
    - Optimize revenue subject to ….

- Inventory Management
  - Contract as many impressions as possible but don’t oversell

- Media Buyer (Arbitrage) (NLP-problem)
  - Talks to publisher
  - Determine publisher mix for network
    - Optimize publisher mix subject to constraints

- Multiobjective optimization
What is Linear Programming

• **Linear programming (LP) =**
  - Linear Algebra + inequalities + optimization (minimize or maximize)

• **LP is a technique for optimization of a linear objective function, subject to linear equality and linear inequality constraints.**
  - Informally, linear programming determines the way to achieve the best outcome (such as maximum profit or lowest cost) in a given mathematical model and given some list of requirements represented as linear equations.
  - More formally, given a polytope (for example, a polygon or a polyhedron), and a real-valued affine function
    \[
    f(x_1, x_2, \ldots, x_n) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n + d
    \]
  - defined on this polytope, a linear programming method will find a point in the polytope where this function has the smallest (or largest) value.
Forecast the supply and create demand

• Forecasting is key to enabling future delivery guarantees

• Segment publisher media, audience
  – Segment media by criteria that Advertisers use to target,
  – Accurately estimate inventory \( \Rightarrow \) price/cost forecasts becomes possible.
  – This enable future delivery guarantees with sustainable economics.

• Leverage optimization theory to answer sales questions
  – How do I sell this future inventory in a optimal manner?
  – Should I sell this batch of impressions based upon targeting criteria X (e.g., Sporty person) OR Y (Sporty person, 18-25, San Francisco Bay Area)

  • [Both criteria apply to this batch; clearly criteria Y commands a greater CPM]
  • What is my predicted traffic for criteria Y in November, 2010?
Frame as a Supply and Demand Problem

- **Supply-side (Publisher)**
  - Partition traffic into targeting segments (language that advertiser’s use to describe desired audience)
  - Create volume predictions of each possible segment using past traffic to publisher and third party data partners

- **Demand-side (Advertisers)**
  - Advertisers want their ads to be shown to user’s that have certain targeting criteria

- **Match demand with supply to maximize revenue**
  - Using linear programming (LP)
  - Frame as a transportation problem in LP
Transportation Problem Description

• A transportation problem basically deals with the problem, which aims to find the best way to fulfill the demand of n demand points using the capacities of m supply points.

• While trying to find the best way, generally a variable cost of shipping the product from one supply point to a demand point or a similar constraint should be taken into consideration.
Maximize Revenue: Ad Allocation Example

<table>
<thead>
<tr>
<th>Supply From</th>
<th>Demand</th>
</tr>
</thead>
<tbody>
<tr>
<td>AudienceZone 1 (e.g., Sports, 18-25, San Francisco)</td>
<td>Volume_{ij} Revenu_{ij}</td>
</tr>
<tr>
<td>AudienceZone 2</td>
<td>…</td>
</tr>
<tr>
<td>AudienceZone 3</td>
<td>…</td>
</tr>
<tr>
<td>Demand Contracted PageViews</td>
<td>45</td>
</tr>
</tbody>
</table>

- Use Linear Programming to generate suggested Ad Sales schedule; This schedule shows how much volume is available in each audience zone and how it should be allocated to advertiser targeting criteria to maximize publisher revenue.
- See next slide for example

Given this Transportation Tableau generate the ad display schedule (explore R’s lp_solve)
Maximize Revenue: Ad Allocation Example

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>PredictedSupply PageViews</th>
</tr>
</thead>
<tbody>
<tr>
<td>AudienceZone 1 (e.g., Sports, 18-25, San Francisco)</td>
<td>Volume^{11}<em>{ij} Revenue^{11}</em>{ij}</td>
<td>Volume^{ij}<em>{ij} Revenue^{ij}</em>{ij}</td>
</tr>
<tr>
<td>AudienceZone 2</td>
<td>…</td>
<td>…</td>
</tr>
<tr>
<td>AudienceZone 3</td>
<td>……</td>
<td>Volume^{ij}<em>{ij} Revenue^{ij}</em>{ij}</td>
</tr>
<tr>
<td>Demand Contracted PageViews</td>
<td>45</td>
<td>20</td>
</tr>
</tbody>
</table>

Use Linear Programming to generate suggested Ad Sales schedule; This schedule shows how much volume is available in each audience zone and how it should be allocated to advertiser targeting criteria to maximize publisher revenue. E.g., This schedule shows how much of Audience zone 1 should be sold to advertisers who want to target sport people; this optimized volume, Volume^{11}_{ij} (based on a global picture) will generate a corresponding Revenue^{11}_{ij}.
Maximize Revenue: Using Linear Program

\[ X_{ij} = \text{number of impressions supplied from publisher Audience zone } i \text{ to advertiser (demand) point } j \]

\[
\text{maximize } \sum_{i=1}^{i=m} \sum_{j=1}^{j=n} \text{revenue}_{ij} * Volume_{ij}
\]

subject to

\[
\sum_{j=1}^{j=n} Volume_{ij} \leq \text{supply}_i (i = 1, 2, \ldots, m) \quad (\text{Demand Constraints})
\]

\[
\sum_{i=1}^{i=m} Volume_{ij} \leq \text{demand}_j (j = 1, 2, \ldots, n) \quad (\text{Supply Constraints})
\]

\[ Volume_{ij} \geq 0 (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n) \quad (\text{NonNeg Constraints}) \]
Optimal Map of Supply onto Demand =$$$

• Maximize profit (or some proxy for profit) given limited audience and much advertiser demand.
• Many algos available
  – Simplex algorithm (most popular)
    • Searches for an optimal solution by moving from one basic solution to another, along the edges of the feasible polygon, in direction of cost decrease (Graphically, moves from corner to corner)
  – Interior Point Methods (more recent)
    • Approaches the situation through the interior of the convex polygon
    • Affine Scaling
    • Log Barrier Methods
    • Primal-dual methods
Balanced Transportation Problem

• If Total supply equals to total demand, the problem is said to be a balanced transportation problem:

\[
\sum_{i=1}^{i=m} Si = \sum_{j=1}^{j=n} dj
\]

• Balancing a TP if total supply exceeds total demand
  • If total supply exceeds total demand, we can balance the problem by adding *dummy* demand point. Since shipments to the dummy demand point are not real, they are assigned a cost of zero.
Typical Forward Markets Problems

- **Allocation of Ads to Publisher real estate**
  - Optimize *revenue* subject to ....

- **Inventory Management**
  - Contract as many impressions as possible but don’t oversell
Scheduling Web Advertisements

- Predictive Clustering + Linear Programming = Web Advertisement Scheduler
  - Partition the world of “webpages X users X Ads” as it is sparse
  - Schedule which ads get displayed

- Limited context to show ads
- Many advertisers want their ads shown and are willing to pay

- Maximize profit (or some proxy for profit) given limited real estate (contexts) and many ads.
Sample Problem

Before describing the formalization, we first show an example which helps illustrate the problem and the need for an LP solution. Assume that the accurately estimated numbers of page views for combinations of attribute values (afternoon, sports), (afternoon, not sports), (not afternoon, sports) and (not afternoon, not sports) are 10,000, 10,000, 5,000 and 5,000, respectively. Also assume that there are three ads for each of which 10,000 impressions have been promised, and that the accurately estimated click-through rates of these ads for the combinations of attribute values are as shown in Table 1.

<table>
<thead>
<tr>
<th>Time of day</th>
<th>Page category</th>
<th>Number of page views</th>
<th>Click-through rate (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>ad 1</td>
</tr>
<tr>
<td>afternoon</td>
<td>sports</td>
<td>10,000</td>
<td>2.2</td>
</tr>
<tr>
<td>afternoon</td>
<td>not sports</td>
<td>10,000</td>
<td>2.2</td>
</tr>
<tr>
<td>not afternoon</td>
<td>sports</td>
<td>5,000</td>
<td>2.2</td>
</tr>
<tr>
<td>not afternoon</td>
<td>not sports</td>
<td>5,000</td>
<td>2.2</td>
</tr>
</tbody>
</table>

http://www.research.ibm.com/people/n/nabe/JECR05-NA.pdf
Greedy vs Random Vs LP

- Assume that page views for all combinations of attribute values occur randomly.
- The greedy strategy always selects ad 1 for the first 10,000 page views, ad 2 for the second 10,000 page views and ad 3 for the last 10,000 page views, because \((\text{the click-through rate of ad 1}) > (\text{the click-through rate of ad 2}) > (\text{the click-through rate of ad 3})\) holds for all combinations of attribute values.
  - As a result, we find that the actual click-through rates for ad 1, ad 2 and ad 3 are 2.2%, 1.76 ...% and 1.33 ...%,
  - the total click-through rate for all ads is 1.76%, which is the same rate as what would be obtained by the random selection strategy.
- According to the optimal display schedule in the LP model, click-through rate is 2.1%

<table>
<thead>
<tr>
<th>CTR Ad1</th>
<th>CTR Ad2</th>
<th>CTR Ad3</th>
<th>AvgAdCtr</th>
<th>proportion of impressions</th>
<th>sumproduct (CTRAd*proportion)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.2</td>
<td>1.1</td>
<td>1</td>
<td>1.433333</td>
<td>0.333333</td>
<td>1.766667</td>
</tr>
<tr>
<td>2.2</td>
<td>2.1</td>
<td>1</td>
<td>1.766667</td>
<td>0.333333</td>
<td>1.766667</td>
</tr>
<tr>
<td>2.2</td>
<td>2.1</td>
<td>2</td>
<td>2.1</td>
<td>0.166667</td>
<td>2.1</td>
</tr>
<tr>
<td>2.2</td>
<td>2.1</td>
<td>2</td>
<td>2.1</td>
<td>0.166667</td>
<td>2.1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CTR Ad2</th>
<th>proportion of impressions</th>
<th>sumproduct(CTRAd2*proportion2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>0.333333</td>
<td>1.766667</td>
</tr>
<tr>
<td>2.1</td>
<td>0.333333</td>
<td>1.766667</td>
</tr>
<tr>
<td>2.1</td>
<td>0.166667</td>
<td>2.1</td>
</tr>
<tr>
<td>2.1</td>
<td>0.166667</td>
<td>2.1</td>
</tr>
</tbody>
</table>
Optimal LP Strategy for Example

• In the above case, according to the optimal display schedule in the LP model, ad 1 is always selected for (afternoon, sports), ad 2 for (afternoon, not sports) and ad 3, otherwise.

• The total click-through rate of this optimal schedule is 2.1%
  – and the actual click-through rates for ad 1, ad 2 and ad 3 are 2.2%, 2.1% and 2.0%, respectively.

• Both greedy and random selection strategy have a CTR of 1.76%,
Maximize Revenue: Ad Allocation Example

Use Linear Programming to generate suggested Ad Sales schedule; This schedule shows how much volume is available in each audience zone and how it should be allocated to advertiser targeting criteria to maximize publisher revenue. E.g., This schedule shows how much of Audience zone 1 should be sold to advertisers who want to target sport people; this optimized volume, Volume_{11} (based on a global picture) will generate a corresponding Revenue_{11}.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>AudienceZone 1 (e.g., Sports, 18-25, San Francisco)</td>
<td>Volume^{11} Revenue_{11}</td>
<td>Volume^{ij} Revenue_{ij}</td>
<td>Volume^{ij} Revenue_{ij}</td>
<td>Volume^{ij} Revenue_{ij}</td>
<td>35</td>
</tr>
<tr>
<td>AudienceZone 2</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>50</td>
</tr>
<tr>
<td>AudienceZone 3</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>15</td>
</tr>
<tr>
<td>Demand Contracted PageViews</td>
<td>45</td>
<td>20</td>
<td>30</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>
Maximize Revenue: Using Linear Program

Determine $Volume_{ij}$

- (number of impressions supplied from Audience zone $i$ to advertiser (demand) point $j$)

$$\text{maximize } \sum_{i=1}^{m} \sum_{j=1}^{n} \text{revenue}_{ij} * Volume_{ij}$$

subject to

$$\sum_{j=1}^{n} Volume_{ij} \leq \text{supply}_i (i = 1,2,\ldots,m)$$

$$\sum_{i=1}^{m} Volume_{ij} \leq \text{demand}_j (j = 1,2,\ldots,n)$$

$$Volume_{ij} \geq 0 (i = 1,2,\ldots,m; j = 1,2,\ldots,n)$$

(Demand Constraints)

(Supply Constraints)

(NonNeg Constraints)
Partition using a predictive clustering

• Segmentation is a key step in guaranteed markets
• Partition “webpages X users X Ads” into zones of self-similarity
• (using page, user, Ad and CTR-based variables) Vs (page, user, Ad)

[Chickering et al. 2001]
Results at msnbc.com

• 1.5 Million impress/Day, Dec 1998

<table>
<thead>
<tr>
<th>Cluster source</th>
<th>Lift</th>
</tr>
</thead>
<tbody>
<tr>
<td>Predictive clusters</td>
<td>38%</td>
</tr>
<tr>
<td>Standard clusters</td>
<td>0%</td>
</tr>
<tr>
<td>Hand-assigned clusters</td>
<td>18%</td>
</tr>
</tbody>
</table>

• Cluster users by the pages they visit on the site, the ads they clicked on and their Ad CTRs.
• How about content-based clustering?

[Chickering et al. 2001]
How about agglomerative segmentation?

$$LL(\theta_P) = \sum_{Z \in P} \sum_{j=1}^{m} - \left( C_{Z,j} \log P(\text{click}|Z, j) \right)$$

$$+ \left( D_{Z,j} - C_{Z,j} \right) \log \left( 1 - P(\text{click}|Z, j) \right)$$

where $D_{Z,j}$ is the number of displays (or impressions) for the cluster ad pair $Z, j$ and $C_{Z,j}$ is the number of clicks observed for the same pair. It is well known and straightforward to verify that this is minimized by letting $P(\text{click}|Z, j) = C_{Z,j}/D_{Z,j}$, so the minimum minus log likelihood for a given partition $P$ is given as follows.

$$LL(\theta_P) = \sum_{Z \in P} \sum_{j=1}^{m} - \left( C_{Z,j} \log \frac{C_{Z,j}}{D_{Z,j}} + \left( D_{Z,j} - C_{Z,j} \right) \log \frac{D_{Z,j} - C_{Z,j}}{D_{Z,j}} \right)$$

The penalty term, according to AIC, is the number of free (probability) parameters in a model, and is simply

$$PT(P) = m|P|.$$ 

We let $I(P)$ denote $I(\{C_{Z,j}/D_{Z,j}: Z \in P, j = 1, \ldots, m\})$. Now, the minimization of $I(\theta_P)$ is reduced to that of $I(P)$.

**Greedy heuristic to search the best $P$; then get the click-through rate**
Results for Nakamura, Abe

- Simulation Results
  - 32 Ads, 128 serving contexts (reduced to 32 clusters)

(a) Cumulative click rate

(b) Instantaneous click rate

Modified Interior Point Alg.
Adapting LP for “important” Advertisers

example, wherein 10,000 impressions each have been promised for two ads. Assume that the accurately estimated click-through rates (%) of these ads for combination 1 and 2 of attribute values are as shown in the following table.

<table>
<thead>
<tr>
<th></th>
<th>ad 1</th>
<th>ad 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Combination 1 of attribute values</td>
<td>4.0</td>
<td>2.5</td>
</tr>
<tr>
<td>Combination 2 of attribute values</td>
<td>2.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Then, the optimal solution is the one that always displays ad 1 for combination 1 and ad 2 for combination 2. With this solution, ad 1 will have a high click-through rate while ad 2 suffers from having a low click-through rate of 1.0. This can be a problem. For example, if the advertiser of ad 2 is more important than the advertiser of ad 1, this problem can be dealt with, to some extent, by introducing the ‘degree of importance’ $g_j$ for each ad $j$. The default value of $g_j$ is 1.0, and a greater value indicates greater importance being assigned to ad $j$. We modify the objective function (1) in linear programming formulation, by the following modification, which incorporates the degrees of importance:

$$
\sum_{i=1}^{n} \sum_{j=1}^{m} g_j c_{i,j} k_i d_{i,j}.
$$

(5)

LP is flexible: Add constraints
Forward Markets

• **Allocation of Ads to Publisher real estate**
  – Give ads play in network
    • Optimize *revenue* subject to ….

• **Inventory Management**
  – Contract as many impressions as possible but don’t oversell
Problem 2: Ad allocation problem

- Ad agencies wish to contract as many ad impressions as possible to earn more revenue.
- But overselling is dangerous. So they need to grasp how many sellable impressions remain.
- In case 1, 8000 sellable impression remain for afternoon constraint, since at least 2000 views in (afternoon, sports) are needed for the contract of sports constraint.
- The calculation of the remaining sellable impressions for a certain constraint $t$ should consider contracts for other constraints which overlap constraint $t$.
- $t$ (how many impressions remain the target afternoon (as opposed to afternoon only))

How many page views can I sell for a publisher zone?

Ad allocation problem

Should only overlapping constraints be considered?

- Case 2 says NO!

- The sellable impressions for business constraint is 8000, not 10000. The sports-constraint contract indirectly affects it, though they don’t overlap.

- τ is the business constraint (8000 possible pageviews)

$$\begin{array}{c}
6 & 4 & 2 & 4 & 6 \\
6 & 4 & 2 & 2 & 0 \text{ (after Alloc of 8,000)} \\
6 & 2 & 0 & 0 & 0 \text{ (after alloc of 6,000)}
\end{array}$$
Ad allocation problem

t – the constraint where we want to maximize sellable impressions;

\( i (=1,\ldots,n) \) – other constraints; \( N_i \) – \# of impressions contracted for constraint \( i \);

\( j (=1,\ldots,m) \) – attribute sets; \( P_j \) – \# of page views for attribute \( j \);

\( S = \{(i, j) \colon \text{constraint } i \text{ contains attribute } j\} \); \( Q = \{j \colon \text{attribute } j \text{ contained by constraint } t\} \)

\( X_{ij} \) – number of scheduled impressions for \((i, j)\);

dummy constraint \( N_{n+1} = \sum_{j=1}^{m} P_j \)

dummy attribute set \( P_{m+1} = \sum_{i=1}^{n} N_i \)
Frame as Transportation Problem:

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>AdZone1</td>
</tr>
<tr>
<td></td>
<td>.. Adzone e_j...</td>
</tr>
<tr>
<td></td>
<td>Adzone M.</td>
</tr>
<tr>
<td>AdContract 1</td>
<td>d_{ij}</td>
</tr>
<tr>
<td></td>
<td>d_{ij}</td>
</tr>
<tr>
<td></td>
<td>d_{ij}</td>
</tr>
<tr>
<td></td>
<td>d_{ij}</td>
</tr>
<tr>
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</tr>
<tr>
<td></td>
<td>PageViews</td>
</tr>
<tr>
<td></td>
<td>35</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>Demand Contracted</td>
</tr>
<tr>
<td>DummyAdContract (n+1)</td>
<td>d_{ij}</td>
</tr>
<tr>
<td></td>
<td>d_{ij}</td>
</tr>
<tr>
<td></td>
<td>d_{ij}</td>
</tr>
<tr>
<td></td>
<td>d_{ij}</td>
</tr>
<tr>
<td></td>
<td>15</td>
</tr>
<tr>
<td>Supply PageViews</td>
<td>45</td>
</tr>
<tr>
<td></td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>5</td>
</tr>
</tbody>
</table>

Use LP to calculate how much many page views can I sell for a publisher zone \( t \) (e.g. Business in the example in the paper)?
Ad allocation problem

Problem 2. Minimize
\[ \sum_{j \in Q} \sum_{i: (i,j) \in S, i \neq n+1} x_{i,j} + \sum_{i: (i,m+1) \in S, i \neq n+1} 2x_{i,m+1} \]

under the conditions
\[ \sum_{j: (i,j) \in S} x_{i,j} = N_i \quad \text{for } i = 1, \ldots, n + 1, \]
\[ \sum_{i: (i,j) \in S} x_{i,j} = P_j \quad \text{for } j = 1, \ldots, m + 1, \]
\[ x_{i,j} \geq 0 \quad \text{for } (i, j) \in S. \]

- **Minimize of allocation (do not over commit)**
- **The first term is the cost of impressions for other contracts that are scheduled for page views overlapping with constraint t.**
- **The second term is the overflow cost that charges for the impressions scheduled for the page views of dummy attribute set. It prevents the overselling for other constraints.**
Problem 2. Minimize

$$\sum_{j \in Q} i: (i,j) \in S, i \neq n+1 x_{i,j} + \sum_{i: (i,m+1) \in S, i \neq n+1} 2x_{i,m+1}$$

- Problem 2 is an instance of linear programming. It can be solved efficiently.
- Let the solution be $x_{ij} = a_{ij}$, then the # of sellable impression for constraint $t$ is $\Sigma_{j \in Q} a_{n+1,j} - N_t$
Allocate Impressions to Inventory

Problem 2. Minimize

\[
\sum_{j \in Q} \sum_{i: (i,j) \in S, i \neq n+1} x_{i,j} + \sum_{i: (i,n+1) \in S, i \neq n+1} 2x_{i,m+1}
\]

- **Problem 2 is an instance of linear programming.**

Cost 0: from dummy (n+1) to attribute side (assign constraint side back) does not influence remaining sellable impressions
Cost 1: shaded area, e.g., contract 6000; assign 2000 2000 to overlapping area; remaining impressions in business is decreased by 2000
Cost 2: over selling of other constraints; for sports the contracted impression is 8,000
• End of Lecture
Guidelines for Homework

- Please provide code, graphs and comments in a Word or PDF report. Don’t forget to put your name, email and date of submission on each report. Please follow the Springer LNCS style (templates for Word and Latex are available at
  - http://www.springer.com/computer/lncs?SGWID=0-164-6-793341-0
  - I.e., pretend you are writing a conference paper (at in format)
- Please provide R code in a separate file .R file and embed the code also in your answers along with the graphs and tables. Please comment your code so that I or anybody else can understand it and please cross reference code with problem numbers and descriptions. Please label each figure and table appropriately.
- Please name files as follows: TIM206-2013-HWK-Week01-StudentLastName.R, .doc, .pdf etc..
- Please create a separate driver function for each exercise or exercise part (and comment!) E.g., hw1-Question3.1.1 = function() {.....}
- If you have questions please raise them in class or via email or during office hours if requested
- Homework is due on Wednesday, of the following week by 7PM.
- Please submit your homework by email to: James.Shanahan@gmail.com and Shanahan@soe.ucsc.edu, and jgrahamsf541@gmail.com with the subject “TIM 206 Winter 2013 Homework 4”

• Have fun!
Homework

- Exercises in H&L Book
  - 6-1-1
  - 6-1-3
  - 6-1-5
  - 6-1-7
  - 6-2-1
  - 6-3-1
  - 6-5-1
  - 6-6-1
  - 7-1-1
  - 8-1-2

- HINT: where possible use IOR tutor or R to solve and plot your answers
• End of Homework